Plan

- Recap on linear acoustics
  - Linearized conservation equations
  - Near- & far-field
  - Acoustical compactness
- Acoustical energy and flow-acoustic resonances
- Green’s function and integral solutions
- Introduction to aeroacoustical analogies
  - Lighthill’s analogy
    - M^8 law
    - Choice of the acoustical variable
  - Curle’s analogy and non-compact sources
    - Aeolian tone
  - Ffowcs Williams & Hawkings analogy
    - Application to fan noise
Continuity equation

- For an infinitesimal fluid particle:

\[
\frac{1}{\rho} \frac{D \rho}{D t} = -\nabla \cdot \mathbf{v} - (\text{dilatation rate})
\]

\[
\frac{D \rho}{D t} = \frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \nabla) \rho
\]
Momentum equation

\[ \rho \frac{Dv}{Dt} = -\nabla \Pi + f \]

External body forces applied to the fluid particle

External stresses applied to the fluid particle:
\[ \Pi_{ij} = \rho \delta_{ij} - \sigma_{ij} \]

Hydrostatic pressure

Viscous stresses

\[ \rho \frac{Dv}{Dt} = -\nabla p + \nabla \sigma + f \]
Linearization

- Continuity and momentum equations:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = Q_m \quad \quad \rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \nabla \sigma + \mathbf{f}
\]

- Perturbations = deviations with respect to uniform and stagnant fluid:

\[
\rho = \rho_0 + \rho' \\
p = p_0 + p' \\
\mathbf{v} = \mathbf{v}_0 + \mathbf{v}' = \mathbf{v}'
\]

- At first order, the continuity and momentum equations become:

\[
\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}' = Q_m \\
\rho_0 \frac{\partial \mathbf{v}'}{\partial t} = -\nabla p' + \nabla \cdot \sigma' + \mathbf{f}
\]
Acoustic sources

- Eliminate \( \mathbf{v}' \) from the linearized conservation equations:

\[
\frac{\partial}{\partial t} \left\{ \frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}' = Q_m \right\}
\]

\[-\nabla \cdot \left\{ \rho_0 \frac{\partial \mathbf{v}'}{\partial t} = -\nabla p' + \nabla \cdot \mathbf{\sigma}' + \mathbf{f} \right\}\]

\[
\frac{\partial^2 \rho'}{\partial t^2} - \nabla^2 p' = -\nabla \cdot \mathbf{f} - \nabla \cdot (\nabla \cdot \mathbf{\sigma}') + \frac{\partial Q_m}{\partial t}
\]

more unknowns than equations…
Linearized constitutive equation

- Equation of state: \( p = p(\rho, s) \)

- Perturbation: \( p' = \left( \frac{\partial p}{\partial \rho} \right)_s \rho' + \left( \frac{\partial p}{\partial s} \right)_\rho s' \)

- Definition of the speed of sound: \( c_0^2 \equiv \left( \frac{\partial p}{\partial \rho} \right)_s \)

\[
p' = c_0^2 \rho' + \left( \frac{\partial p}{\partial s} \right)_\rho s'
\]
Sources of sound

\[
\frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = -\nabla \cdot \mathbf{f} - \nabla \cdot (\nabla \cdot \mathbf{\sigma}') + \frac{1}{c_0^2} \left( \frac{\partial p}{\partial s} \right)_\rho \frac{\partial^2 s'}{\partial t^2} + \frac{\partial Q_m}{\partial t}
\]

Mass source is used as model for entropy production. We assume iso-kinetic injection (no momentum source) and isentropic process.
Monopoles, dipoles, quadrupoles

- Monopole = pulsating sphere, jumping in a boat
  - Physically: unsteady combustion, pipe exhaust, vocal folds, ...

- Dipole = oscillating sphere, playing with a ball in a boat
  - Less efficient than monopole
  - Physically: unsteady forces

- Quadrupole = deforming sphere without change of volume nor net force, fighting in a boat!
  - Less efficient than dipole
  - Physically: turbulence
Spherical waves

- Homogeneous wave propagation equation in spherical coordinates:

\[
\frac{1}{c_0^2} \frac{\partial^2 (rp')}{\partial t^2} - \frac{\partial^2 (rp')}{\partial r^2} = 0
\]

- Same solution as in 1D using \( rp' \) as unknown:

\[
rp' = f \left( t - \frac{r}{c_0} \right) + g \left( t + \frac{r}{c_0} \right)
\]

- Outgoing wave

- Incoming wave

- In frequency domain:

\[
p' = \frac{A}{r} \exp \left( i\omega \left( t - \frac{r}{c_0} \right) \right) = \frac{A}{r} \exp \left( i (\omega t - kr) \right)
\]
Far-field and near-field

- Linearized momentum equation in spherical coordinates:

\[
\rho_0 \frac{\partial v'_r}{\partial t} = -\frac{\partial p'}{\partial r}
\quad \Rightarrow \quad v'_r = \frac{p'}{\rho_0 c_0} \left( 1 + \frac{1}{ikr} \right)
\]

Specific impedance: \( k = \omega/c_0 \)

- Two regimes:

  - Far-field: 
    \[
    \lim_{kr \to \infty} v'_r = \frac{p'}{\rho_0 c_0} \propto \frac{1}{r}
    \]
    Plane wave behaviour

  - Near-field:
    \[
    \lim_{kr \to 0} v'_r = \frac{p'}{i\omega \rho_0 r} \propto \frac{1}{r^2}
    \]
    Locally incompressible flow
Acoustical compactness

- Upon normalization using the length scale $L$ and the time scale $\tau$: $\tilde{t} \equiv t/\tau \quad \tilde{x}_i \equiv x_i/L$

  the wave propagation equation

  \[
  \frac{\partial^2 \phi'}{\partial \tilde{t}^2} - c_0^2 \frac{\partial^2 \phi'}{\partial \tilde{x}_i^2} = 0
  \]

  becomes:

  \[
  \frac{\partial^2 \phi'}{\partial \tilde{x}_i^2} = \left( \frac{L}{c_0 \tau} \right)^2 \frac{\partial^2 \phi'}{\partial \tilde{t}^2} = He^2 \frac{\partial^2 \phi'}{\partial \tilde{t}^2}
  \]

  with Helmholtz number

  \[
  He \equiv \frac{L}{c_0 \tau} = \frac{\omega L}{c_0} = \frac{2\pi L}{\lambda} = k L
  \]

- Compact region: $He \ll 1 \Rightarrow \nabla^2 \phi' = 0 \quad \text{Laplace equation}$

At low Helmholtz numbers, i.e. in a compact region, the wave propagation equation reduces to the Laplace equation, describing an incompressible potential flow

Corollary: an incompressible potential flow model solves the “acoustics problem” in a compact region
Acoustical energy

- Manipulating linearized conservation equations:

\[
\frac{p'}{\rho_0} \left\{ \frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}' = Q_m \right\} \\
+ \mathbf{v}' \cdot \left\{ \rho_0 \frac{\partial \mathbf{v}'}{\partial t} = -\nabla p' + \nabla \cdot \mathbf{\sigma}' + \mathbf{f} \right\}
\]

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} \rho_0 (\mathbf{v}')^2 + \frac{1}{2} \frac{p'^2}{\rho_0 c_0^2} \right) + \nabla \cdot (p' \mathbf{v}') = \mathbf{v}' \cdot \mathbf{f} + \frac{p'}{\rho_0 c_0^2} \left( \frac{\partial p}{\partial s} \right)_\rho \frac{\partial s'}{\partial t} + \frac{p' Q_m}{\rho_0}
\]
**Acoustical energy and intensity**

\[
\frac{\partial E}{\partial t} + \nabla \cdot \mathbf{I} = \mathbf{v}' \cdot \mathbf{f} + \frac{p'}{\rho_0 c_0^2} \left( \frac{\partial p}{\partial s} \right)_\rho \frac{\partial s'}{\partial t} + \frac{p' Q_m}{\rho_0}
\]

- **Forces (e.g. vibrating walls)**
- **Volume source**
- **Entropic processes (e.g. combustion)**

\[
E = \frac{1}{2} \rho_0 (\mathbf{v}')^2 + \frac{1}{2} \frac{p'^2}{\rho_0 c_0^2}
\]

Acoustic energy

\[
\mathbf{I} = p' \mathbf{v}'
\]

Acoustic intensity
Integral formulation for steady harmonic oscillations

\[
\langle P \rangle = \iiint_S \langle \mathbf{I} \cdot \mathbf{n} \rangle \, dS = \iiint_V \langle \mathbf{v}' \cdot \mathbf{f} + \frac{p'Q_m}{\rho_0} \rangle \, dV
\]
Free field Green’s function in 3 dimensions

- Inhomogeneous wave equation:
  \[ \frac{\partial^2 G}{\partial t^2} - c_0^2 \nabla^2 G = \delta(x - y) \delta(t - \tau) \]

- Solution:
  \[ G(x, t|y, \tau) = \frac{\delta\left(t - \tau - \frac{|x - y|}{c_0}\right)}{4\pi c_0^2 |x - y|} \]

- Retarded (emission) time:
  \[ \tau^* = t - \frac{|x - y|}{c_0} \]

- Important properties:
  - Dirac function \( \rightarrow \) convenient to obtain an integral solution
  - Reciprocity:
    \[ G(x, t|y, \tau) = G(y, -\tau|x, -t) \]
Solution of the wave equation based on Green’s function

\[ \left\{ \frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \nabla^2 \rho' = q(x, t) \right\} \times G, \iiint_V, \int_{t_0}^t \]

\[ - \left\{ \frac{\partial^2 G'}{\partial t^2} - c_0^2 \nabla^2 G = \delta(x - y) \delta(t - \tau) \right\} \times \rho', \iiint_V, \int_{t_0}^t \]

\[ \rho'(x, t) = \int_{t_0}^t \iiint_V q(y, \tau) G(x, t|y, \tau) \, d^3y \, d\tau \]

\[ + \int_{t_0}^t \iiint_V \left( \rho'(y, \tau) \frac{\partial^2 G}{\partial \tau^2} - G \frac{\partial^2 \rho'(y, \tau)}{\partial \tau^2} \right) \, d^3y \, d\tau \]

\[ - c_0^2 \int_{t_0}^t \iiint_V \left( \rho'(y, \tau) \frac{\partial^2 G}{\partial y_i^2} - G \frac{\partial^2 \rho'(y, \tau)}{\partial y_i^2} \right) \, d^3y \, d\tau \]
Integral solution of the wave equation

- Integrating by parts:
  \[ \rho'(\mathbf{x}, t) = \int_{t_0}^{t} \int_{V} \int_{V} q(\mathbf{y}, \tau) \, G(\mathbf{x}, \tau | \mathbf{y}, \tau) \, d^3y \, d\tau \]
  \[ - \left[ \int_{V} \int_{V} \left( \rho'(\mathbf{y}, \tau) \frac{\partial G}{\partial \tau} - G \frac{\partial \rho'(\mathbf{y}, \tau)}{\partial \tau} \right) \, d^3y \right]_{\tau=t_0} \]
  \[ - c_0^2 \int_{t_0}^{t} \int_{S} \int_{S} \left( \rho'(\mathbf{y}, \tau) \frac{\partial G}{\partial y_i} - G \frac{\partial \rho'(\mathbf{y}, \tau)}{\partial y_i} \right) \, n_i \, d^2y \, d\tau \]

- Further simplifications:
  - Silent initial conditions, causality
  - No solid surface, OR: non-vibrating surfaces and tailored Green’s function

\[ \rho'(\mathbf{x}, t) = \int_{t_0}^{t} \int_{V} \int_{V} q(\mathbf{y}, \tau) \, G(\mathbf{x}, \tau | \mathbf{y}, \tau) \, d^3y \, d\tau \]

Having an integral formulation improves the numerical stability of the prediction when detailed flow data (e.g. LES) are available, and otherwise permits deriving scaling laws!
Other Green’s functions

- In a few cases: analytical Green’s functions
  - Infinite planes: image sources (semi-anechoic environment)
  - Semi-infinite plane (trailing edge noise)
  - Infinite straight ducts: rectangular, cylindrical, annular

- In other cases: semi-analytical Green’s functions
  - Compact (low-frequency) Green’s functions (Howe)
  - Wiener-Hopf technique, Schwarzchild’s technique (TE-LE backscattering, Roger)
  - Slowly-varying duct (Rienstra)

- In all other cases: numerical Green’s functions
  - Low-frequency techniques
    - Finite Element Methods, Boundary Element Methods
  - High-frequency techniques
    - Ray-tracing methods, Statistical Energy Analysis
  - Mid-frequency techniques
    - Multigrid techniques, fast multipole BEM, ...
Summary

- Assuming small amplitude acoustic perturbations, the equations of fluid motion can be linearized and used to derive a wave equation for these perturbations.
- The relationship between the perturbations are given by the linearized momentum equation and the linearized constitutive equation:

\[ \rho_0 \frac{\partial \mathbf{v}'}{\partial t} = -\nabla p' \]

\[ p' = c_0^2 \rho' + \left( \frac{\partial p}{\partial s} \right)_\rho s' \]

- In the linear approximation, the sources of the acoustic field can be due to
  - Unsteady mass injection or entropy fluctuations \( \rightarrow \) monopolar character.
  - Non-uniform forces \( \rightarrow \) dipolar character.
  - Fluctuating viscous stresses (and, later, Reynolds stresses) \( \rightarrow \) quadrupolar character.
- Each of these sources has a different radiation efficiency in free field.
- The sound radiation is determined by the source and the impedance which it experiences!
- An integral formulation of the wave equation can be obtained using Green’s functions, which enhances the numerical robustness of the prediction.
Aeroacoustic analogies: why?

- Acoustic field = part of the flow field → most straightforward approach: Computational AeroAcoustics (CAA)
  - But: at low Mach numbers: orders of magnitude of difference between
    - Length scales: \( \lambda_{ac} = \frac{L_{turb}}{M} \)
    - Magnitudes: \( O(M^4) \) of the flow energy radiates into the far field

\[ p' = 4.4934739 \text{ Pa} \]

- High order schemes needed to capture acoustic propagation
- Numerical cost of a direct CAA scales with \( Re^2 M^{-4} \) for a Large Eddy Simulation
- Specific issues related to CFD discretisation techniques applied to acoustics
  - Dissipation and dispersion errors
  - Initial and boundary conditions
Lighthill’s aeroacoustical analogy: concept

- The problem of sound produced by a turbulent flow is, from the listener’s point of view, analogous to a problem of propagation in a uniform medium at rest in which equivalent sources are placed.

- Wave propagation region: linear wave operator applies
  \[
  \frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \frac{\partial^2 \rho'}{\partial x_i^2} = 0
  \]

- Turbulent region: fluid mechanics equations apply
  \[
  \begin{align*}
  \frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x_i} &= 0 \\
  \frac{\partial \rho v_i}{\partial t} + \frac{\partial \rho v_i v_j}{\partial x_j} &= -\frac{\partial \Pi_{ij}}{\partial x_j} \\
  \Pi_{ij} &= p \delta_{ij} - \sigma_{ij}
  \end{align*}
  \]
Lighthill’s analogy: formal derivation

\[ \frac{\partial}{\partial t} \left\{ \frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x_i} \right\} = 0 \]  
Continuity

\[- \frac{\partial}{\partial x_i} \left\{ \frac{\partial \rho v_i}{\partial t} + \frac{\partial \rho v_i v_j}{\partial x_j} \right\} = - \frac{\partial \Pi_{ij}}{\partial x_j} \]  
Momentum

\[ \frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2 (\rho v_i v_j - \sigma_{ij})}{\partial x_i \partial x_j} + \frac{\partial^2 p}{\partial x_i^2} \]

\[ \frac{\partial^2 \rho}{\partial t^2} - c_0^2 \frac{\partial^2 \rho}{\partial x_i^2} = \frac{\partial^2 (\rho v_i v_j - \sigma_{ij})}{\partial x_i \partial x_j} + \frac{\partial^2 (p - c_0^2 \rho)}{\partial x_i^2} \]
Lighthill’s aeroacoustical analogy: reference state

- Reformulation of fluid mechanics equations, and use of arbitrary speed $c_0$:
  \[
  \frac{\partial^2 \rho}{\partial t^2} - c_0^2 \frac{\partial^2 \rho}{\partial x_i^2} = \frac{\partial^2 (\rho v_i v_j - \sigma_{ij})}{\partial x_i \partial x_j} + \frac{\partial^2 (p - c_0^2 \rho)}{\partial x_i^2}
  \]

- Definition of a reference state:
  \[
  \rho' \equiv \rho - \rho_0 \\
  p' \equiv p - p_0 \\
  v_i' \equiv v_i
  \]

- Aeroacoustical analogy:
  \[
  \frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \frac{\partial^2 \rho'}{\partial x_i^2} = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}
  \]
  with \( T_{ij} = \rho v_i v_j + (p' - c_0^2 \rho') \delta_{ij} - \sigma_{ij} \)

\textit{Exact... and perfectly useless!}
Sound produced by free isothermal turbulent flows at low Mach number

- Solution using Green’s fct

\[ \rho'(x, t) = \int_{-\infty}^{t} \int \int \int_V \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} G \, d^3 y \, d\tau - c_0^2 \int_{-\infty}^{t} \int \int_{\partial V} \left( \rho' \frac{\partial G}{\partial y_i} - G \frac{\partial \rho'}{\partial y_i} \right) n_i \, d^2 y \, d\tau \]

- Purpose: simplify the RHS

- High Reynolds number
- Isentropic
- Low Mach number

\[ T_{ij} = \rho v_i v_j + (p - c_0^2 \rho') \delta_{ij} - \partial^2 \]

- Using free field Green’s fct

\[ G_0(t, x | \tau, y) = \frac{\delta(t - \tau - |x - y|/c_0)}{4\pi c_0^2|x - y|} \]

\[ \rho'(x, t) = \frac{\partial^2}{\partial x_i \partial x_j} \int \int \int_V \left[ \frac{\rho_0 v_i v_j}{4\pi c_0^2|x - y|} \right] d^3 y \]

**Quadrupolar source**

\[ t^* = t - \frac{|x - y|}{c_0} \]
**Lighthill’s $M^8$ law**

- **Integral solution:**
  \[
  \rho'(x, t) = \frac{\partial^2}{\partial x_i \partial x_j} \int \int \int_V \left[ \frac{\rho_0 v_i v_j}{4\pi c_0^2 |x - y|} \right] d^3y
  \]

  \[
  t^* = t - \frac{|x - y|}{c_0}
  \]

- **Scaling law:**
  - Acoustic scale: \(x \propto \lambda = c_0 / f\)
  - Flow time scale: \(D / U_0\)
  - Spatial derivative: \(U_0 / (c_0D)\)

- **Acoustical power:**
  \[
  W = \frac{4\pi |x|^2 p'^2}{\rho_0 c_0} \propto \rho_0 c_0^3 D^2 M^8
  \]
Choice of the aeroacoustical variable

- Manipulating the mass and momentum equations yields:

\[
\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial x_i \partial x_j} (\rho v_i v_j - \sigma_{ij}) + \frac{\partial^2 p}{\partial x_i^2}
\]

- From there, two choices are possible for the acoustical variable:
  - Acoustical density perturbation:

\[
\frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \frac{\partial^2 \rho'}{\partial x_i^2} = \frac{\partial^2}{\partial x_i \partial x_j} (\rho v_i v_j - \sigma_{ij}) + \frac{\partial^2}{\partial x_i^2} (p' - c_0^2 \rho')
\]

  - Isentropic noise generation

  - Acoustical pressure perturbation:

\[
\frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x_i^2} = \frac{\partial^2}{\partial x_i \partial x_j} (\rho v_i v_j - \sigma_{ij}) + \frac{\partial^2}{\partial t^2} \left( \frac{p'}{c_0^2} - \rho' \right)
\]

  - Combustion noise
Curle’s analogy: fixed rigid bodies

- **Lighthill’s aeroacoustical analogy:**
  \[
  \frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \frac{\partial^2 \rho'}{\partial x_i^2} = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}
  \]

- **Integral solution using Green’s function**
  \[
  \rho'(x, t) = \int_{-\infty}^{t} \int_{-\infty}^{t} \int_{V} \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} G \ d^3y \ d\tau \quad \text{incident field}
  \]
  \[
  - c_0^2 \int_{-\infty}^{t} \int_{\partial V} \left( \rho' \frac{\partial G}{\partial y_i} - G \frac{\partial \rho'}{\partial y_i} \right) n_i \ d^2y \ d\tau \quad \text{scattered field}
  \]

- **Partial integration of source integral**
  \[
  \int_{-\infty}^{t} \int_{-\infty}^{t} \int_{V} \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} G \ d^3y \ d\tau = \int_{-\infty}^{t} \int_{-\infty}^{t} \int_{V} T_{ij} \frac{\partial^2 G}{\partial y_i \partial y_j} \ d^3y \ d\tau
  \]
  \[
  + \int_{-\infty}^{t} \int_{\partial V} \left\{ \left( - \frac{\partial \rho v_i}{\partial \tau} - c_0^2 \frac{\partial \rho'}{\partial y_i} \right) G - \left( \rho v_i v_j + (p' - c_0^2 \rho') \delta_{ij} + \sigma_{ij} \right) \frac{\partial G}{\partial y_j} \right\} n_i \ d^2y \ d\tau
  \]

- **Curle’s analogy: uses free field Green’s function**
  \[
  G_0(t, x|\tau, y) = \frac{\delta(t - \tau - |x - y|/c_0)}{4\pi c_0^2 |x - y|}
  \]
  \[
  \rho'(x, t) = \frac{\partial^2}{\partial x_i \partial x_j} \int_{V} \int_{V} \left[ \frac{T_{ij}}{4\pi c_0^2 |x - y|} \right] \ d^3y - \frac{\partial}{\partial x_i} \int_{\partial V} \left[ \frac{p'n_i}{4\pi c_0^2 |x - y|} \right] \ d^2y
  \]
If you know the multipolar character of your source...

... use the corresponding Green’s function!
A popular formulation for industrial applications

- Curle's formulation is quite powerful
  - It enforces the correct radiation pattern of each source component: 
    \[ P_{\text{quadru}} / P_{\text{dipo}} \sim M^2 \]
  - At low Mach numbers, dipolar contribution dominates the quadrupolar one for compact sources
  - Surface scalar \((\rho')\) data are much less demanding in memory than volumetric, tensorial \((T_{ij})\) data
  - Surface mesh often available from design stage

\[
4\pi c_0^2 \rho'(x, t) = \frac{\partial^2}{\partial x_i \partial x_j} \int \int \int_V \left[ \frac{T_{ij}}{|x - y|} \right] d^3y + \frac{\partial}{\partial x_i} \int \int_{\partial V} \left[ \frac{\rho' n_i}{|x - y|} \right] d^2y
\]

- **Quadrupole, \(W \propto M^8\) in free field**
- **Dipole, \(W \propto M^6\) in free field**

- BUT: tricky implementation for non-compact geometries...
The hybrid approach from a practical viewpoint

- The computation of flow is decoupled from the computation of sound
  - Acoustic prediction: post-processing of source field data

- Fundamental assumption: one-way coupling
  - Unsteady flow produces sound and affects its propagation
  - BUT: sound waves do not affect flow field significantly
  - Principal application of the hybrid approach: flows at low Mach numbers

- Usable simulation tools for the flow description:
  - Reynolds Averaged Navier-Stokes (RANS) solver → time-averaged data (SNGR to reconstruct transient)
  - Unsteady RANS → unsteady, but only large scale
  - Large Eddy Simulation (LES), Detached Eddy Simulation (DES) → unsteady, broadband turbulence (up to grid & scheme cut-off frequency)

- Low-Mach number applications
  - Incompressible LES / DES solvers to reduce CPU cost
  - Careful interpretation of the flow data in aeroacoustical analogy

http://www.lmfa.ec-lyon.fr/recherche/turbo
Ffowcs Williams and Hawkings analogy: moving bodies

- Use of generalized functions to account for body motion
- Body motion trajectory described by function $F(x,y,z,t)$:

$$F = 0$$
$$|\nabla F| = 1$$

$F > 0$

$F < 0$

- Heaviside and Dirac functions properties:

$$H(F) = 0 \quad \text{inside solid body}$$
$$H(F) = 1 \quad \text{in fluid region}$$

$$\nabla (H(F)) = \delta(F) \nabla F$$
$$\frac{\partial}{\partial t} (H(F)) = \delta(F) \frac{\partial F}{\partial t}$$
Conservative equations and analogy using generalized functions

- Equations of conservation of mass and momentum can be rewritten as:

\[
\frac{\partial (\rho' H)}{\partial t} + \frac{\partial (\rho v_i H)}{\partial x_i} = \rho_0 (\mathbf{v} \cdot \nabla F) \delta(F) = \rho_0 V_n \delta(F)
\]

\[
\frac{\partial (\rho v_i H)}{\partial t} + \frac{\partial}{\partial x_j} \left[(\rho v_i v_j + p \delta_{ij} + \sigma_{ij}) H\right] = (p \delta_{ij} + \sigma_{ij}) \nabla F \delta(F)
\]

- The analogy becomes:

\[
\frac{\partial^2 (\rho' H)}{\partial t^2} - c_0^2 \frac{\partial^2 (\rho' H)}{\partial x_i^2} = \frac{\partial^2}{\partial x_i \partial x_j} (T_{ij} H) - \frac{\partial}{\partial x_i} \left[(p \delta_{ij} + \sigma_{ij}) \nabla F \delta(F)\right] + \frac{\partial}{\partial t} (\rho_0 V_n \delta(F))
\]
Integral solution

- Using the free-field Green's function:

\[ \rho'(x, t) = \frac{\partial^2}{\partial x_i \partial x_j} \int_{-\infty}^{t} \int_{-\infty}^{t} \int_{\mathcal{V}} \frac{\delta(t - \tau - |x - y|/c_0)}{4\pi c_0^2 |x - y|} T_{ij}(y, \tau) \, d^3 y \, d\tau \]

\[ - \frac{\partial}{\partial x_i} \int_{-\infty}^{t} \int_{\partial \mathcal{V}} \frac{\delta(t - \tau - |x - y|/c_0)}{4\pi c_0^2 |x - y|} [p \delta_{ij} + \sigma_{ij}] (y, \tau) n_j \, d^2 y \, d\tau \]

\[ + \frac{\partial}{\partial t} \int_{-\infty}^{t} \int_{\partial \mathcal{V}} \frac{\delta(t - \tau - |x - y|/c_0)}{4\pi c_0^2 |x - y|} \rho_0 V_n(y, \tau) \, d^2 y \, d\tau \]

- More practical to have the source described in a moving coordinate system \( \eta \) attached to the body.

- Retarded time equation:

\[ g(t^*, t, x, \eta) \equiv t - t^* - \frac{|x - y(\eta, t^*)|}{c_0} = 0 \]

- Dirac function property:

\[ \int_{-\infty}^{\infty} \delta(h(\xi)) \, f(\xi) \, d\xi = \sum_i \frac{f(\xi_i)}{|h'(\xi_i)|}, \quad h(\xi_i) = 0 \]
Doppler effects

- Applied to the quadrupolar component:

\[
\int_{-\infty}^{t} \iiint_{V_\eta} \frac{\delta(g(\tau, t, x, \eta))}{4\pi c_0^2 |x - y(\eta, \tau)|} T_{ij} \, d^3\eta \, d\tau = \frac{1}{4\pi c_0^2} \iiint_{V_\eta} \left[ \frac{T_{ij}}{R \left| 1 - \frac{\mathbf{M} \cdot \mathbf{R}}{R} \right|} \right] d^3\eta
\]

\[\mathbf{M} : \text{vector Mach number} \quad 1 - \frac{\mathbf{M} \cdot \mathbf{R}}{R} : \text{Doppler factor}\]

- Final solution:

\[
\rho'(x, t) = \frac{1}{4\pi c_0^2} \frac{\partial^2}{\partial x_i \partial x_j} \iiint_{V_\eta} \left[ \frac{T_{ij}}{R \left| 1 - \frac{\mathbf{M} \cdot \mathbf{R}}{R} \right|} \right] d^3\eta
\]

\[- \frac{1}{4\pi c_0^2} \frac{\partial}{\partial x_i} \iiint_{\partial V_\eta} \left[ \frac{(p \delta_{ij} + \sigma_{ij}) n_j}{R \left| 1 - \frac{\mathbf{M} \cdot \mathbf{R}}{R} \right|} \right] d^2\eta
\]

\[+ \frac{1}{4\pi c_0^2} \frac{\partial}{\partial t} \iiint_{\partial V_\eta} \left[ \frac{\rho_0 V_n}{R \left| 1 - \frac{\mathbf{M} \cdot \mathbf{R}}{R} \right|} \right] d^2\eta
\]

- The flow intrinsic features (flow separation, turbulent transition, ...) are expressed in the frame of reference attached to the moving axes.

- The Doppler effects (convective amplification and frequency shift) are given by the motion of the sources in the fixed coordinate system.
Effect of relative motion

- **Fixed monopole**
- **Monopole convected at \( M = 0.7 \)**

(source: Wikipedia)
Doppler amplification

\[ \Delta \varphi - \frac{1}{c_0^2} \frac{\partial^2 \varphi}{\partial t^2} = S(x, t) \quad S(y, \tau) = -q(\tau) \delta(y - U_0\tau) \]

\[ \varphi(x, t) = \int_{-\infty}^{\infty} \int_V q(\tau) \frac{\delta(\tau - t + |x - y|/c_0)}{4\pi|x - y|} \delta(y - U_0\tau) \, d^3y \, d\tau \]

\[ \int_{-\infty}^{\infty} f(\xi') \delta(g(\xi')) \, d\xi' = \sum_j \frac{f(\xi_j)}{\left| \frac{\partial g}{\partial \xi}(\xi_j) \right|} \]

\[ g(\tau) \equiv \tau - t + |x - y(\tau)|/c_0 = 0 \]

\[ \varphi(x, t) = \frac{1}{4\pi} \sum_j \frac{q(\tau_j)}{R_j \left[ 1 - M_0 \cos \theta_j \right]} \]

Doppler factor: amplitude modulation and directivity
Doppler frequency shift

\[ q(\tau) = \hat{q} e^{i\omega \tau} \]

\[ \varphi(x, t) = \frac{\hat{q}}{4\pi} \frac{e^{i\omega(t-R_1(t)/c_0)}}{R_1(1 - M_0 \cos \theta_1)} \]

\[ \frac{R_1(t)}{c_0} \sim \frac{R_1(t_e)}{c_0} + \frac{1}{c_0} \left. \frac{dR_1}{dt} \right|_{t_e} (t - t_e) + \ldots \]

\[ \frac{dR_1}{dt} = -\frac{M_0 \cos \theta_1}{1 - M_0 \cos \theta_1} \]

Phase factor: \( e^{i\omega t/(1 - M_0 \cos \theta_1(t_e))} \)

Modified frequency: \( \frac{\omega}{1 - M_0 \cos \theta_1(t_e)} \)

**The Doppler factor accounts for all amplitude modulation and frequency shift effects; the source unsteadiness can be described in a local (moving) coordinate system attached to the moving body.**
Thickness, loading, turbulence noise

- Thickness noise:

- Loading noise: steady and unsteady

\[ \frac{\partial}{\partial \tau} \left( \frac{F_i}{D} \right) = \frac{1}{D} \frac{\partial F_i}{\partial \tau} - \frac{F_i}{D^2} \frac{\partial D}{\partial \tau} \]

unsteady motion

\[ D = |1 - M \cdot R/R| \]

- Turbulence noise: same scaling, relative to unsteady loading noise, as for the motion loading noise \( \rightarrow \) negligible as well

At low Mach numbers, and for thin rotor/stator blades/vanes, the unsteady loading noise dominates the sound production.
Loading noise mechanisms

- **Incoming turbulence noise**
- **Trailing edge (self) noise**
- **T-S waves**
- **Vortex shedding**
- **Boundary layer turbulence**

- **Incoming turbulence**
Viscous and potential interactions

Stator vanes: varying inlet velocity angle and magnitude

Rotor blades: varying outlet velocity angle and magnitude
Viscous interaction: tonal and broadband noise
Tonal and broadband noise

Rotating point force

- FW-H analogy in time domain:

\[
\rho'(x, t) = \int_{-\infty}^{t} \int_{S(\tau)} \frac{\partial G}{\partial y_i} f_i \, d^2 y \, d\tau + \int_{-\infty}^{t} \int_{V(\tau)} \frac{\partial^2 G}{\partial y_i \partial y_j} T_{ij} \, d^3 y \, d\tau + \int_{-\infty}^{t} \int_{S(\tau)} \frac{\partial G}{\partial \tau} \rho_0 V_n \, d^2 y \, d\tau
\]

- Using the free field Green’s function, and for a compact source:

\[
\rho'(x, t) \sim -\frac{1}{4\pi c_0^2} \frac{\partial}{\partial x_i} \int_{S(\tau_0)} \left[ \frac{f_i}{R \left| 1 - \mathbf{M} \mathbf{R} / R \right|} \right] \, d^2 \zeta
\]

\[
\sim -\frac{1}{4\pi c_0^2} \frac{\partial}{\partial x_i} \left[ \frac{F_i}{RD} \right]
\]

- In the Fourier domain:

\[
\rho(x, \omega) = \frac{ik}{8\pi^2 c_0^2} \int_{-\infty}^{\infty} \frac{\mathbf{F} \cdot \mathbf{R}}{R^2} \left( 1 + \frac{1}{ikR} \right) e^{-i\omega(\tau + R/c_0)} \, d\tau
\]
Geometrical far-field approximation for a $B$-bladed axial rotor

Constructive interference: sound of the total fan = $B \times$ (sound of a single blade)

Bessel function: modulation of the Doppler frequency shift during blade revolution

$$\rho_{nB} \sim -\frac{iBk_{nB}}{4\pi c_0^2} \frac{e^{-ik_{nB}x}}{x} \sum_{p=-\infty}^{+\infty} e^{-i(nB-p)(\varphi-\pi/2)} J_{-nB+p}(-k_{nB}R_0 \sin \theta) \left[ F_p^{(T)} \cos \theta - \frac{nB-p}{k_{nB}R_0} F_p^{(D)} \right]$$

Sound emitted at BPFHs

Sum over BLHs

Listener distance

Radius where force is applied

Listener azimuthal angle

Listener polar angle

Drag harmonic

Thrust harmonic
Doppler effect $\rightarrow$ summation over the BLHs

$$\rho_{nB} \sim -\frac{iBk_{nB}}{4\pi c_0^2} \frac{e^{-ik_{nB}x}}{x} \sum_{p=-\infty}^{+\infty} e^{-i(nB-p)(\varphi-\pi/2)} J_{-nB+p}(-k_{nB}R_0 \sin \theta) \left[ F_p^{(T)} \cos \theta - \frac{nB - p}{k_{nB}R_0} F_p^{(D)} \right]$$

$B = 4$
Summary

- Aeroacoustical analogies allow extracting a maximum of acoustical information from a given description of the flow field.

- Assuming a decoupling between the sound production and propagation, the analogies provide an explicit integral solution for the acoustical field at the listener position:
  - Improves numerical robustness
  - Permits drawing scaling laws

- **BUT:** one has to make approximations and choices!
  - Acoustical variable (e.g. isothermal noise vs combustion noise)
  - Source term formulation (e.g. Lighthill’s analogy vs Vortex Sound Theory)

- Some formulations make the dominant character of the source appear more explicitly, and allow making useful approximations.
- Without approximations, the analogy is useless!
A few references

- S.W. Rienstra and A. Hirschberg, *An Introduction To Acoustics* (corrections), Report IWDE 01-03 May 2001, revision every year or so...

- And of course: the VKI Lecture Series...