Graduate Program in Acoustics
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Introduction to Aeroacoustics

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Plan

- Recap on linear acoustics
    - Linearized conservation equations
    - Near- & far-field
    - Acoustical compactness
- Acoustical energy and flow-acoustic resonances
- Green's function and integral solutions
- Introduction to aeroacoustical analogies
    - Lighthill's analogy
        - M^8 law
        - Choice of the acoustical variable
    - Curle's analogy and non-compact sources
        - Aeolian tone
    - Ffowcs Williams & Hawkings analogy
        - Application to fan noise

Continuity equation

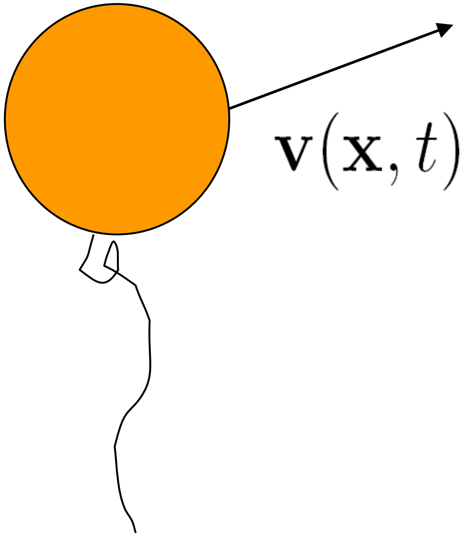
- For an infinitesimal fluid particle:

rate of change of density moving with the fluid particle

– (dilatation rate)

$$\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\nabla\cdot\mathbf{v}$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial\rho}{\partial t} + (\mathbf{v}\cdot\nabla)\rho$$



Momentum equation

Acceleration of the fluid particle

External body forces applied to the fluid particle

$$\rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = -\nabla \mathbf{\Pi} + \mathbf{f}$$

External stresses applied to the fluid particle:

$$\Pi_{ij} = p\,\boldsymbol{\delta}_{ij} - \sigma_{ij}$$

Hydrostatic pressure

Viscous stresses

$$\rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = -\nabla p + \nabla \boldsymbol{\sigma} + \mathbf{f}$$

Linearization

- Continuity and momentum equations:

Introduced for convenience

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = Q_\mathrm{m} \qquad \rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = -\nabla p + \nabla \boldsymbol{\sigma} + \mathbf{f}$$

- Perturbations = deviations with respect to uniform and stagnant fluid:

$$\rho = \rho_0 + \rho'$$
$$p = p_0 + p'$$
$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}' = \mathbf{v}'$$

- At first order, the continuity and momentum equations become:

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}' = Q_\mathrm{m} \qquad \rho_0 \frac{\partial \mathbf{v}'}{\partial t} = -\nabla p' + \nabla \cdot \boldsymbol{\sigma}' + \mathbf{f}$$

Acoustic sources

- Eliminate $\mathbf{v}'$ from the linearized conservation equations:

$$\frac{\partial}{\partial t}\left\{\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}' = Q_m\right\}$$

$$-\nabla \cdot \left\{\rho_0 \frac{\partial \mathbf{v}'}{\partial t} = -\nabla p' + \nabla \cdot \boldsymbol{\sigma}' + \mathbf{f}\right\}$$

$$\Rightarrow \quad \frac{\partial^2 \rho'}{\partial t^2} - \nabla^2 p' = -\nabla \cdot \mathbf{f} - \nabla \cdot (\nabla \cdot \ \boldsymbol{\sigma}') + \frac{\partial Q_m}{\partial t}$$

more unknowns than equations…

Linearized constitutive equation

- Equation of state: $p = p(\rho, s)$
- Perturbation: $p' = \left(\frac{\partial p}{\partial \rho}\right)_s \rho' + \left(\frac{\partial p}{\partial s}\right)_\rho s'$
- Definition of the speed of sound: $c_0^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_s$

$$\Rightarrow \quad p' = c_0^2 \rho' + \left(\frac{\partial p}{\partial s}\right)_\rho s'$$

Sources of sound

D'Alembertian

$$\frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = -\nabla\cdot\mathbf{f} - \nabla\cdot(\nabla\cdot\boldsymbol{\sigma}') + \frac{1}{c_0^2}\left(\frac{\partial p}{\partial s}\right)_\rho \frac{\partial^2 s'}{\partial t^2} + \frac{\partial Q_\mathrm{m}}{\partial t}$$

Non-uniform force field

Fluctuating viscous stresses

Entropy fluctuations

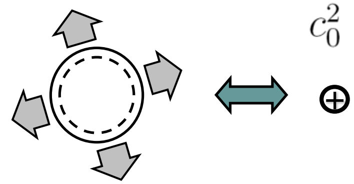
Fluctuating mass injection

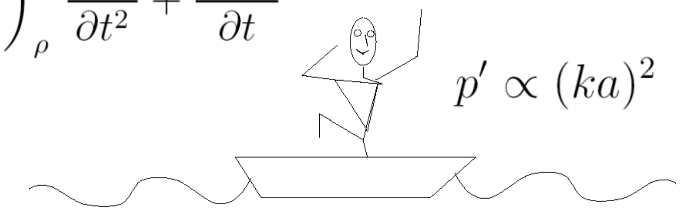
<u>Mass source is used as model for entropy production.</u>
We assume iso-kinetic injection (no momentum source) and isentropic process.

Monopoles, dipoles, quadrupoles

- Monopole = pulsating sphere, jumping in a boat
- Physically: unsteady combustion, pipe exhaust, vocal folds, …

$$\frac{1}{c_0^2}\left(\frac{\partial p}{\partial s}\right)_\rho \frac{\partial^2 s'}{\partial t^2} + \frac{\partial Q_\mathrm{m}}{\partial t}$$

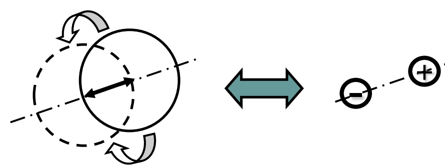


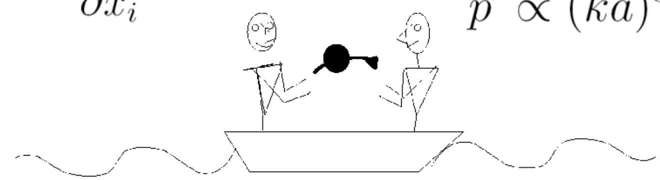


$$p' \propto (ka)^2$$

- Dipole = oscillating sphere, playing with a ball in a boat
- Less efficient than monopole
- Physically: unsteady forces

$$\frac{\partial f_i}{\partial x_i}$$

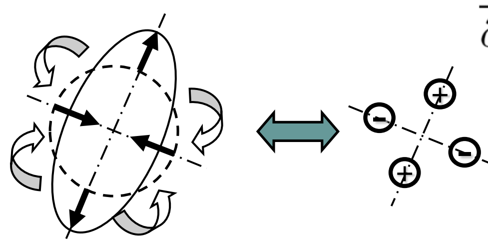


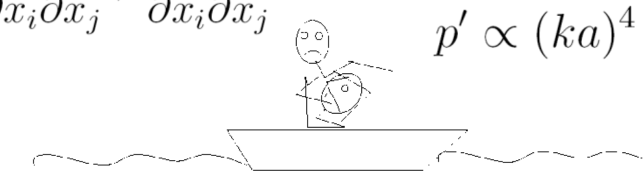


$$p' \propto (ka)^3$$

- Quadrupole = deforming sphere without change of volume nor net force, fighting in a boat!
- Less efficient than dipole
- Physically: turbulence

$$\frac{\partial^2 \sigma_{ij}}{\partial x_i \partial x_j}, \quad \frac{\partial^2 \rho v_i v_j}{\partial x_i \partial x_j}$$





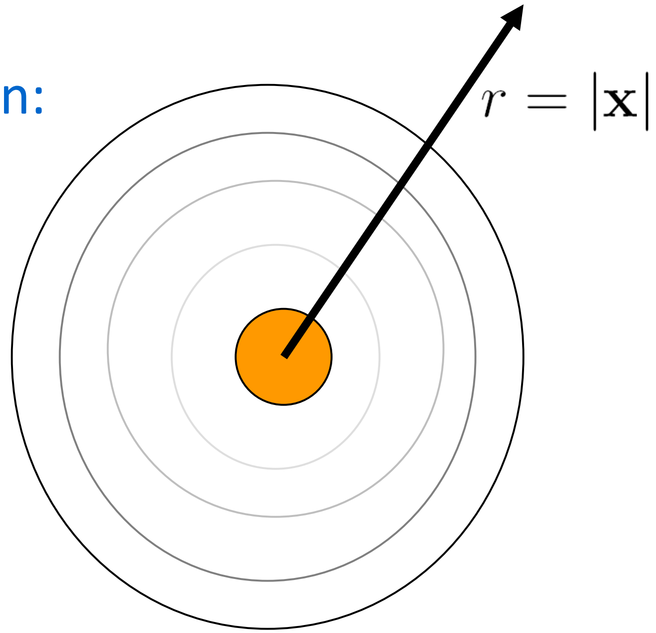
$$p' \propto (ka)^4$$

Spherical waves

- Homogeneous wave propagation equation in spherical coordinates:

$$\frac{1}{c_0^2}\frac{\partial^2(rp')}{\partial t^2} - \frac{\partial^2(rp')}{\partial r^2} = 0$$

- Same solution as in 1D using rp' as unknown:



$$rp' = f\left(t - \frac{r}{c_0}\right) + g\left(t + \frac{r}{c_0}\right)$$

Outgoing wave

Incoming wave

- In frequency domain:

$$p' = \frac{A}{r}\exp\left(\mathrm{i}\omega\left(t - \frac{r}{c_0}\right)\right) = \frac{A}{r}\exp\left(\mathrm{i}\left(\omega t - kr\right)\right)$$

Far-field and near-field

- Linearized momentum equation in spherical coordinates:

$$\rho_0 \frac{\partial v_r'}{\partial t} = -\frac{\partial p'}{\partial r} \quad \Rightarrow \quad v_r' = \frac{p'}{\rho_0 c_0}\left(1 + \frac{1}{\mathrm{i}kr}\right) \qquad k = \omega / c_0$$

Specific impedance

- Two regimes:
  - Far-field: $$\lim_{kr \to \infty} v_r' = \frac{p'}{\rho_0 c_0} \propto \frac{1}{r}$$ Plane wave behaviour
  - Near-field: $$\lim_{kr \to 0} v_r' = \frac{p'}{\mathrm{i}\omega \rho_0 r} \propto \frac{1}{r^2}$$ Locally incompressible flow

Acoustical compactness

- Upon normalization using the length scale *L* and the time scale τ: $\tilde{t} \equiv t/\tau \qquad \tilde{x}_i \equiv x_i/L$

  the wave propagation equation $\dfrac{\partial^2 \varphi'}{\partial t^2} - c_0^2 \dfrac{\partial^2 \varphi'}{\partial x_i^2} = 0$

  becomes: $$\frac{\partial^2 \varphi'}{\partial \tilde{x}_i^2} = \left(\frac{L}{c_0 \tau}\right)^2 \frac{\partial^2 \varphi'}{\partial \tilde{t}^2} = He^2 \frac{\partial^2 \varphi'}{\partial \tilde{t}^2}$$

  with Helmholtz number $He \equiv \dfrac{L}{c_0 \tau} = \dfrac{\omega L}{c_0} = \dfrac{2\pi L}{\lambda} = kL$

- Compact region: $He \ll 1 \Rightarrow \nabla^2 \varphi' = 0$ Laplace equation

⇨ ***At low Helmholtz numbers, i.e. in a compact region, the wave propagation equation reduces to the Laplace equation, describing an <u>incompressible potential</u> flow***

⇨ ***Corollary: an incompressible potential flow model solves the "acoustics problem" in a compact region***

Acoustical energy

- Manipulating linearized conservation equations:

$$\frac{p'}{\rho_0}\left\{\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}' = Q_\mathrm{m}\right\}$$

$$+\mathbf{v}' \cdot \left\{\rho_0 \frac{\partial \mathbf{v}'}{\partial t} = -\nabla p' + \nabla \cdot \boldsymbol{\sigma}' + \mathbf{f}\right\}$$

$$\Downarrow$$

$$\frac{\partial}{\partial t}\underbrace{\left(\frac{1}{2}\rho_0 \left(\mathbf{v}'\right)^2 + \frac{1}{2}\frac{p'^2}{\rho_0 c_0^2}\right)}_{E} + \nabla \cdot \underbrace{\left(p'\mathbf{v}'\right)}_{\mathbf{I}} = \mathbf{v}' \cdot \mathbf{f} + \frac{p'}{\rho_0 c_0^2}\left(\frac{\partial p}{\partial s}\right)_\rho \frac{\partial s'}{\partial t} + \frac{p' Q_\mathrm{m}}{\rho_0}$$

Acoustical energy and intensity

Forces (e.g. vibrating walls)

Volume source

$$\frac{\partial E}{\partial t} + \nabla \cdot \mathbf{I} = \mathbf{v}' \cdot \mathbf{f} + \frac{p'}{\rho_0 c_0^2} \left(\frac{\partial p}{\partial s} \right)_\rho \frac{\partial s'}{\partial t} + \frac{p' Q_\mathrm{m}}{\rho_0}$$

Entropic processes
(e.g. combustion)

$$E = \frac{1}{2} \rho_0 \left(\mathbf{v}' \right)^2 + \frac{1}{2} \frac{p'^2}{\rho_0 c_0^2}$$

Acoustic energy

$$\mathbf{I} = p' \mathbf{v}'$$

Acoustic intensity

Integral formulation for steady harmonic oscillations

Power generated by unsteady forces

Power generated by unsteady volume injection (or entropy fluctuations)

$$\langle P \rangle = \iint_S \langle \mathbf{I} \cdot \mathbf{n} \rangle \, \mathrm{d}S = \iiint_V \langle \mathbf{v}' \cdot \mathbf{f} + \frac{p' Q_\mathrm{m}}{\rho_0} \rangle \, \mathrm{d}V$$

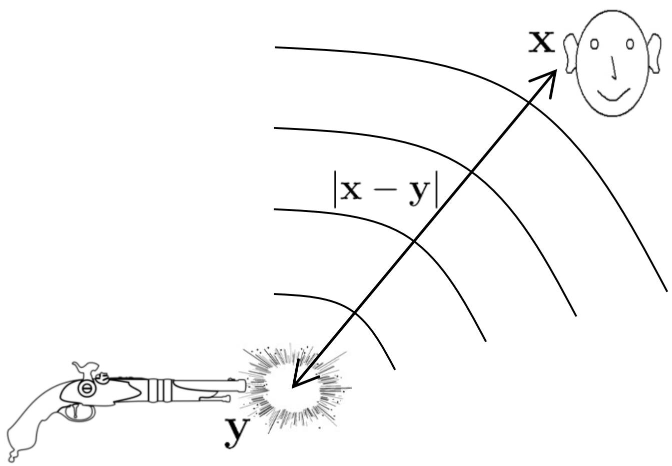
Power, averaged over one cycle

Power entering or leaving the domain through its boundaries

Free field Green's function in 3 dimensions

- Inhomogeneous wave equation: $$\frac{\partial^2 G}{\partial t^2} - c_0^2 \nabla^2 G = \delta(\mathbf{x}-\mathbf{y})\,\delta(t-\tau)$$

- Solution: $$G\left(\mathbf{x},t|\mathbf{y},\tau\right) = \frac{\delta\left(t-\tau-\dfrac{|\mathbf{x}-\mathbf{y}|}{c_0}\right)}{4\pi c_0^2\,|\mathbf{x}-\mathbf{y}|}$$



- Retarded (emission) time: $$\tau^* = t - \frac{|\mathbf{x}-\mathbf{y}|}{c_0}$$

- Important properties:
  - Dirac function → convenient to obtain an integral solution
  - Reciprocity: $G\left(\mathbf{x},t|\mathbf{y},\tau\right) = G\left(\mathbf{y},-\tau|\mathbf{x},-t\right)$

Solution of the wave equation based on Green's function

$$\left\{ \frac{\partial^2 \rho'}{\partial t^2} - c_0^2\,\nabla^2 \rho' = q(\mathbf{x}, t) \right\} \quad \times\, G,\ \iiint_V,\ \int_{t_0}^{t}$$

$$-\left\{ \frac{\partial^2 G}{\partial t^2} - c_0^2\,\nabla^2 G = \boldsymbol{\delta}(\mathbf{x} - \mathbf{y})\,\boldsymbol{\delta}(t - \tau) \right\} \quad \times\, \rho',\ \iiint_V,\ \int_{t_0}^{t}$$

$$\Longrightarrow \quad \begin{aligned} \rho'(\mathbf{x}, t) = {} & \int_{t_0}^{t} \iiint_V q(\mathbf{y}, \tau)\, G\left(\mathbf{x}, t | \mathbf{y}, \tau\right)\ \mathrm{d}^3\mathbf{y}\mathrm{d}\tau \\ & + \int_{t_0}^{t} \iiint_V \left(\rho'(\mathbf{y}, \tau)\, \frac{\partial^2 G}{\partial \tau^2} - G\, \frac{\partial^2 \rho'(\mathbf{y}, \tau)}{\partial \tau^2} \right)\ \mathrm{d}^3\mathbf{y}\mathrm{d}\tau \\ & - c_0^2 \int_{t_0}^{t} \iiint_V \left(\rho'(\mathbf{y}, \tau)\, \frac{\partial^2 G}{\partial y_i^2} - G\, \frac{\partial^2 \rho'(\mathbf{y}, \tau)}{\partial y_i^2} \right)\ \mathrm{d}^3\mathbf{y}\mathrm{d}\tau \end{aligned}$$

Integral solution of the wave equation

- Integrating by parts: $$\rho'(\mathbf{x},t) = \int_{t_0}^{t}\iiint_V q(\mathbf{y},\tau)\, G\left(\mathbf{x},t|\mathbf{y},\tau\right)\,\mathrm{d}^3\mathbf{y}\mathrm{d}\tau - \left[\iiint_V \left(\rho'(\mathbf{y},\tau)\,\frac{\partial G}{\partial \tau} - G\,\frac{\partial \rho'(\mathbf{y},\tau)}{\partial \tau}\right)\,\mathrm{d}^3\mathbf{y}\right]_{\tau=t_0} - c_0^2\int_{t_0}^{t}\iint_S \left(\rho'(\mathbf{y},\tau)\,\frac{\partial G}{\partial y_i} - G\,\frac{\partial \rho'(\mathbf{y},\tau)}{\partial y_i}\right)\,n_i\,\mathrm{d}^2\mathbf{y}\mathrm{d}\tau$$
- Further simplifications:
  - Silent initial conditions, causality
  - No solid surface, OR: non-vibrating surfaces and tailored Green's fuction

$$\Rightarrow \quad \rho'(\mathbf{x},t) = \int_{t_0}^{t}\iiint_V q(\mathbf{y},\tau)\, G\left(\mathbf{x},t|\mathbf{y},\tau\right)\,\mathrm{d}^3\mathbf{y}\mathrm{d}\tau$$

Having an integral formulation improves the numerical stability of the prediction when detailed flow data (e.g. LES) are available, and otherwise permits deriving scaling laws!

Other Green’s functions

- In a few cases: analytical Green’s functions
  - Infinite planes: image sources (semi-anechoic environment)
  - Semi-infinite plane (trailing edge noise)
  - Infinite straight ducts: rectangular, cylindrical, annular
- In other cases: semi-analytical Green’s functions
  - Compact (low-frequency) Green’s functions (Howe)
  - Wiener-Hopf technique, Schwarzchild’s technique (TE-LE backscattering, Roger)
  - Slowly-varying duct (Rienstra)
- In all other cases: numerical Green’s functions
  - Low-frequency techniques
    - Finite Element Methods, Boundary Element Methods
  - High-frequency techniques
    - Ray-tracing methods, Statistical Energy Analysis
  - Mid-frequency techniques
    - Multigrid techniques, fast multipole BEM, …

Summary

- Assuming small amplitude acoustic perturbations, the equations of fluid motion can be linearized and used to derive a wave equation for these perturbations.
- The relationship between the perturbations are given by the linearized momentum equation and the linearized constitutive equation:

$$\rho_0 \frac{\partial \mathbf{v}'}{\partial t} = -\nabla p' \qquad\qquad p' = c_0^2 \rho' + \left(\frac{\partial p}{\partial s}\right)_\rho s'$$

- In the linear approximation, the sources of the acoustic field can be due to
  - Unsteady mass injection or entropy fluctuations → monopolar character.
  - Non-uniform forces → dipolar character.
  - Fluctuating viscous stresses (and, later, Reynolds stresses) → quadrupolar character.
- Each of these sources has a different radiation efficiency in free field.
- The sound radiation is determined by the source and the impedance which it experiences!
- An integral formulation of the wave equation can be obtained using Green's functions, which enhances the numerical robustness of the prediction.

Aeroacoustic analogies: why ?

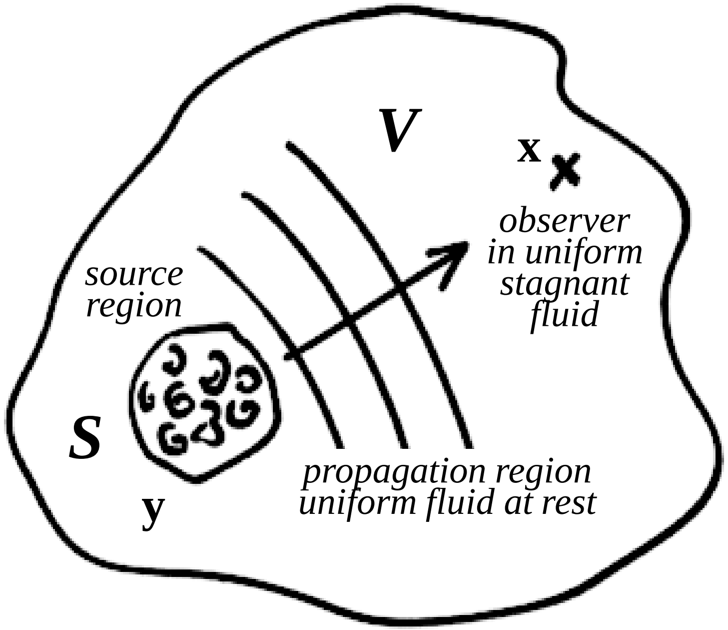
- Acoustic field = part of the flow field → most straightforward approach: Computational AeroAcoustics (CAA)
  - But: at low Mach numbers: orders of magnitude of difference between
    - Length scales: $\lambda_{ac} = L_{turb} / M$
    - Magnitudes: O(M^4) of the flow energy radiates into the far field

$$p' = \underbrace{4.493}_{\text{hydrodynamic field}}\underbrace{4739}_{\text{acoustic field}}\ \text{Pa}$$

  - High order schemes needed to capture acoustic propagation
  - Numerical cost of a direct CAA scales with $Re^2\ M^{-4}$ for a Large Eddy Simulation
  - Specific issues related to CFD discretisation techniques applied to acoustics
    - Dissipation and dispersion errors
    - Initial and boundary conditions

Lighthill's aeroacoustical analogy : concept

- The problem of sound produced by a turbulent flow is, **from the listener's point of view**, analogous to a problem of propagation in a uniform medium at rest in which equivalent sources are placed.



- Wave propagation region: linear wave operator applies

No source

$$\frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \frac{\partial^2 \rho'}{\partial x_i^2} = 0$$

- Turbulent region: fluid mechanics equations apply

mass

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x_i} = 0$$

momentum

$$\frac{\partial \rho v_i}{\partial t} + \frac{\partial \rho v_i v_j}{\partial x_j} = -\frac{\partial \Pi_{ij}}{\partial x_j}$$

$$\Pi_{ij} = p\,\delta_{ij} - \sigma_{ij}$$

Lighthill's analogy: formal derivation

$$\frac{\partial}{\partial t}\left\{\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x_i} = 0\right\}$$

Continuity

$$-\frac{\partial}{\partial x_i}\left\{\frac{\partial \rho v_i}{\partial t} + \frac{\partial \rho v_i v_j}{\partial x_j} = -\frac{\partial \Pi_{ij}}{\partial x_j}\right\}$$

Momentum

$$\Rightarrow \quad \frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2 (\rho v_i v_j - \sigma_{ij})}{\partial x_i \partial x_j} + \frac{\partial^2 p}{\partial x_i^2}$$

$$\frac{\partial^2 \rho}{\partial t^2} - \boxed{c_0^2 \frac{\partial^2 \rho}{\partial x_i^2}} = \frac{\partial^2 (\rho v_i v_j - \sigma_{ij})}{\partial x_i \partial x_j} + \frac{\partial^2 (p - \boxed{c_0^2 \rho})}{\partial x_i^2}$$

Lighthill's aeroacoustical analogy : reference state

- Reformulation of fluid mechanics equations, and use of arbitrary speed c_0 :

$$\frac{\partial^2 \rho}{\partial t^2} - c_0^2 \frac{\partial^2 \rho}{\partial x_i^2} = \frac{\partial^2 (\rho v_i v_j - \sigma_{ij})}{\partial x_i \partial x_j} + \frac{\partial^2 (p - c_0^2 \rho)}{\partial x_i^2}$$

- Definition of a reference state:

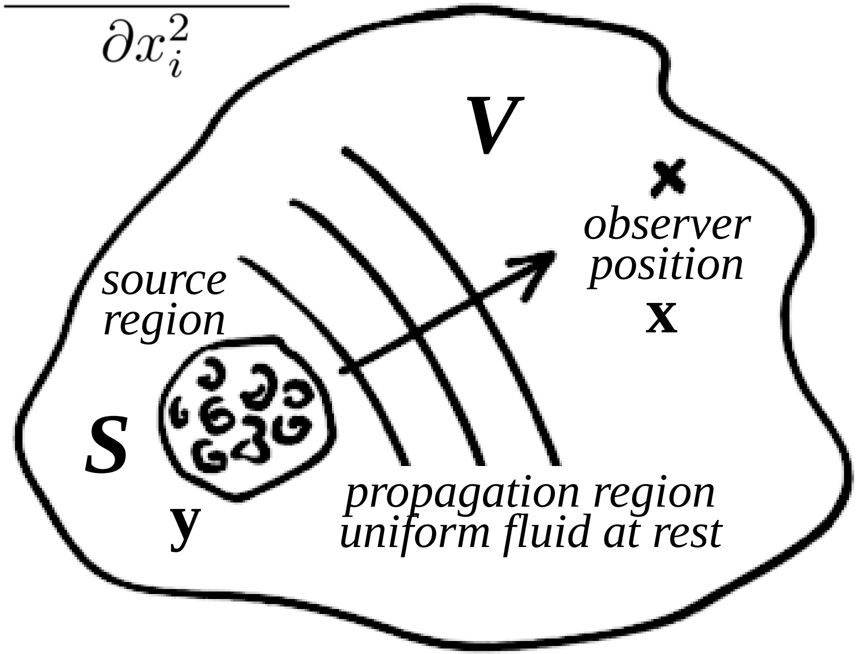
$$\rho' \equiv \rho - \rho_0$$
$$p' \equiv p - p_0$$
$$v_i' \equiv v_i$$

- Aeroacoustical analogy :

$$\frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \frac{\partial^2 \rho'}{\partial x_i^2} = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}$$

with $T_{ij} = \rho v_i v_j + (p' - c_0^2 \rho') \delta_{ij} - \sigma_{ij}$

Lighthill's tensor



Exact… and perfectly useless!

Sound produced by free isothermal turbulent flows at low Mach number

- Solution using Green's fct

$$\rho'(\mathbf{x},t) = \int_{-\infty}^{t} \iiint_V \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} G \, \mathrm{d}^3\mathbf{y}\mathrm{d}\tau \overbrace{- c_0^2 \int_{-\infty}^{t} \iint_{\partial V} \left(\rho' \frac{\partial G}{\partial y_i} - G \frac{\partial \rho'}{\partial y_i} \right) n_i \, \mathrm{d}^2\mathbf{y} \, \mathrm{d}\tau}^{\textit{sound scattering at boundaries}}$$

integral solution

- Purpose: simplify the RHS $\dfrac{\partial^2 T_{ij}}{\partial x_i \partial x_j}$ $\qquad T_{ij} = \rho v_i v_j + \cancel{(p' - c_0^2 \rho')\,\delta_{ij}} - \cancel{\sigma_{ij}}$

  - High Reynolds number
  - Isentropic
  - Low Mach number $\longrightarrow \rho v_i v_j \simeq \rho_0 v_i v_j$

- Using free field Green's fct $G_0(t,\mathbf{x}|\tau,\mathbf{y}) = \dfrac{\delta(t - \tau - |\mathbf{x}-\mathbf{y}|/c_0)}{4\pi c_0^2 |\mathbf{x}-\mathbf{y}|}$

$$\Rightarrow \rho'(\mathbf{x},t) = \frac{\partial^2}{\partial x_i \partial x_j} \iiint_V \left[\frac{\rho_0 v_i v_j}{4\pi c_0^2 |\mathbf{x}-\mathbf{y}|} \right] \mathrm{d}^3\mathbf{y}$$

$$t^* = t - |\mathbf{x}-\mathbf{y}|/c_0$$

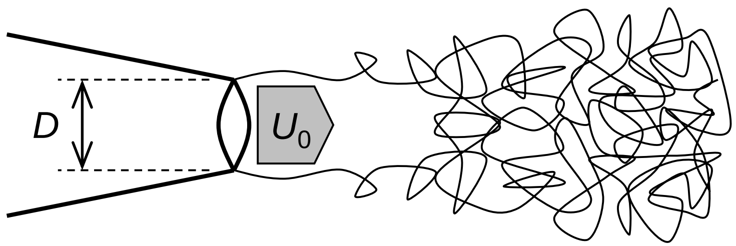
$\boxed{\textit{Quadrupolar source}}$

Lighthill's M^8 law

- Integral solution: $\rho'(\mathbf{x},t) = \dfrac{\partial^2}{\partial x_i \partial x_j} \iiint_V \left[\dfrac{\rho_0 v_i v_j}{4\pi c_0^2 |\mathbf{x}-\mathbf{y}|} \right] \mathrm{d}^3\mathbf{y}$

$$t^* = t - |\mathbf{x}-\mathbf{y}|/c_0$$

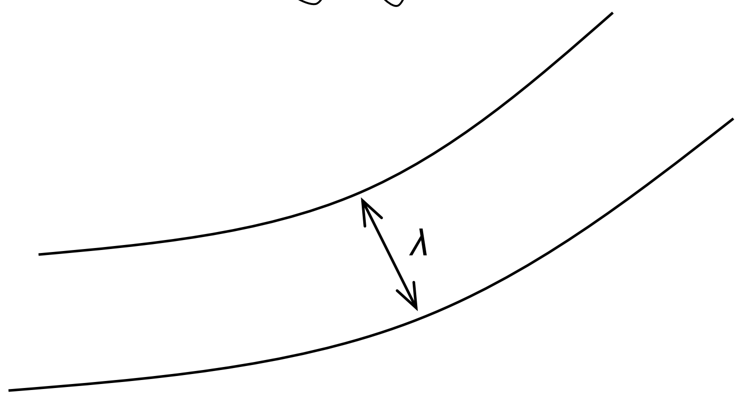
- Scaling law:



Acoustic scale: $x \propto \lambda = c_0/f$

Flow time scale: D/U_0

Spatial derivative: $U_0/(c_0 D)$



$$p' = c_0^2 \rho' \propto \frac{U_0^2}{c_0^2 D^2} \frac{\rho_0 U_0^2 D^3}{|\mathbf{x}|} = \rho_0 c_0^2 M^4 \frac{D}{|\mathbf{x}|}$$

Acoustical power: $W = \dfrac{4\pi |\mathbf{x}|^2 p'^2}{\rho_0 c_0} \propto \rho_0 c_0^3 D^2 M^8$

Choice of the aeroacoustical variable

- Manipulating the mass and momentum equations yields:

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial x_i \partial x_j}\left(\rho v_i v_j - \sigma_{ij}\right) + \frac{\partial^2 p}{\partial x_i^2}$$

- From there, two choices are possible for the acoustical variable:
  - Acoustical density perturbation:

$$\frac{\partial^2 \rho'}{\partial t^2} \boxed{- c_0^2 \frac{\partial^2 \rho'}{\partial x_i^2}} = \frac{\partial^2}{\partial x_i \partial x_j}\left(\rho v_i v_j - \sigma_{ij}\right) + \frac{\partial^2}{\partial x_i^2}\left(p' \boxed{- c_0^2 \rho'}\right)$$

Isentropic noise generation

  - Acoustical pressure perturbation:

$$\boxed{\frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2}} - \frac{\partial^2 p'}{\partial x_i^2} = \frac{\partial^2}{\partial x_i \partial x_j}\left(\rho v_i v_j - \sigma_{ij}\right) + \frac{\partial^2}{\partial t^2}\left(\boxed{\frac{p'}{c_0^2}} - \rho'\right)$$

Combustion noise

Curle’s analogy: fixed rigid bodies

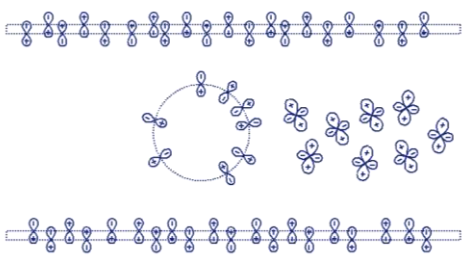
- Lighthill’s aeroacoustical analogy: $\dfrac{\partial^2 \rho'}{\partial t^2} - c_0^2 \dfrac{\partial^2 \rho'}{\partial x_i^2} = \dfrac{\partial^2 T_{ij}}{\partial x_i \partial x_j}$
- Integral solution using Green’s function

$$\rho'(\mathbf{x},t) = \int_{-\infty}^{t} \iiint_V \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} G \, \mathrm{d}^3\mathbf{y} \mathrm{d}\tau$$

incident field

$$- c_0^2 \int_{-\infty}^{t} \iint_{\partial V} \left(\rho' \frac{\partial G}{\partial y_i} - G \frac{\partial \rho'}{\partial y_i} \right) n_i \, \mathrm{d}^2\mathbf{y} \, \mathrm{d}\tau$$

scattered field



- Partial integration of source integral

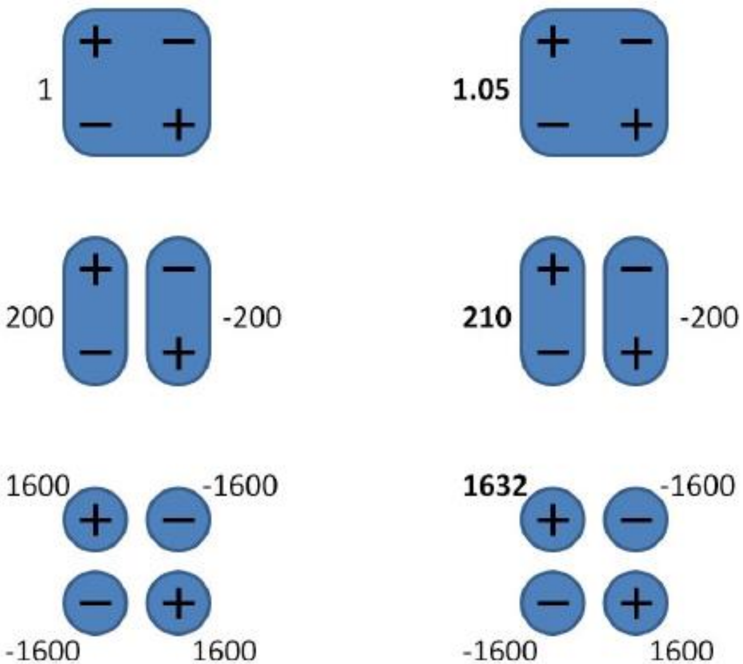
$$\int_{-\infty}^{t} \iiint_V \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} G \, \mathrm{d}^3\mathbf{y} \mathrm{d}\tau = \int_{-\infty}^{t} \iiint_V T_{ij} \frac{\partial^2 G}{\partial y_i \partial y_j} \mathrm{d}^3\mathbf{y} \, \mathrm{d}\tau$$

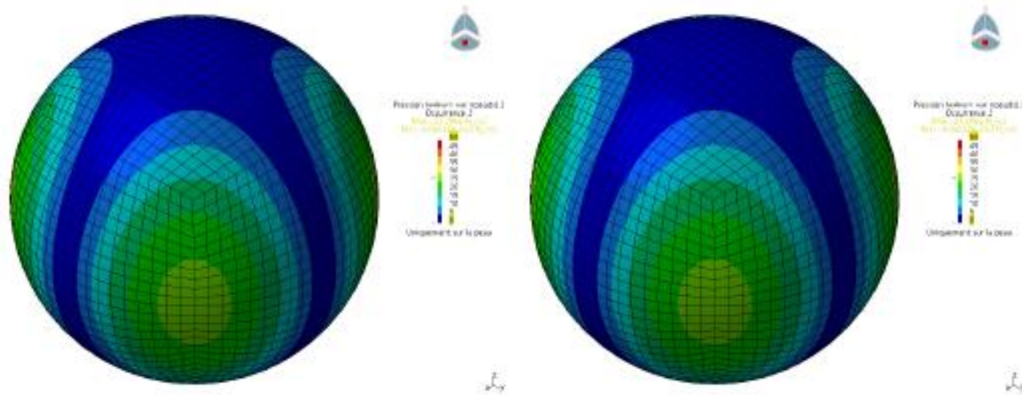
$$+ \int_{-\infty}^{t} \iint_{\partial V} \left\{ \left(- \frac{\partial \rho v_i}{\partial \tau} - \underline{c_0^2 \frac{\partial \rho'}{\partial y_i}} \right) G - \left(\rho v_i v_j + (p' - \underline{c_0^2 \rho'}) \, \delta_{ij} + \sigma_{ij} \right) \frac{\partial G}{\partial y_j} \right\} n_i \, \mathrm{d}^2\mathbf{y} \, \mathrm{d}\tau$$

- Curle’s analogy: uses free field Green’s function $G_0(t,\mathbf{x}|\tau,\mathbf{y}) = \dfrac{\delta(t-\tau-|\mathbf{x}-\mathbf{y}|/c_0)}{4\pi c_0^2 |\mathbf{x}-\mathbf{y}|}$

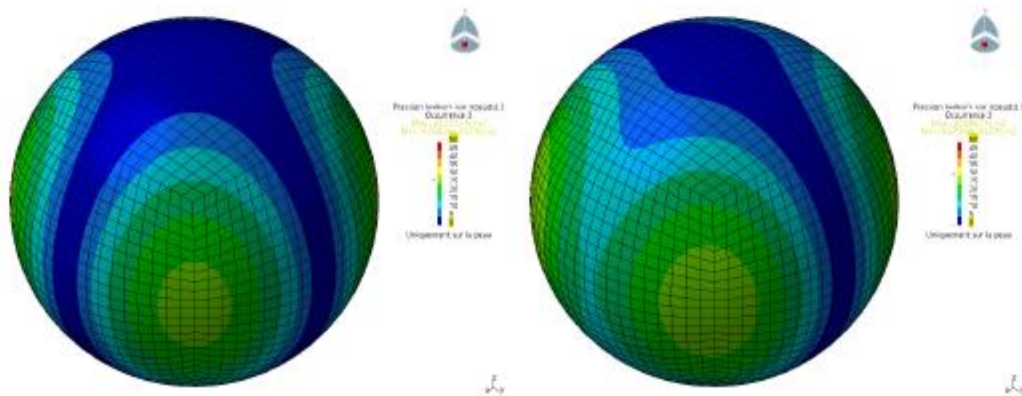
$$\Rightarrow \quad \rho'(\mathbf{x},t) = \frac{\partial^2}{\partial x_i \partial x_j} \iiint_V \left[\frac{T_{ij}}{4\pi c_0^2 |\mathbf{x}-\mathbf{y}|} \right] \mathrm{d}^3\mathbf{y} - \frac{\partial}{\partial x_i} \iint_{\partial V} \left[\frac{p' n_i}{4\pi c_0^2 |\mathbf{x}-\mathbf{y}|} \right] \mathrm{d}^2\mathbf{y}$$

If you know the multipolar character of your source…

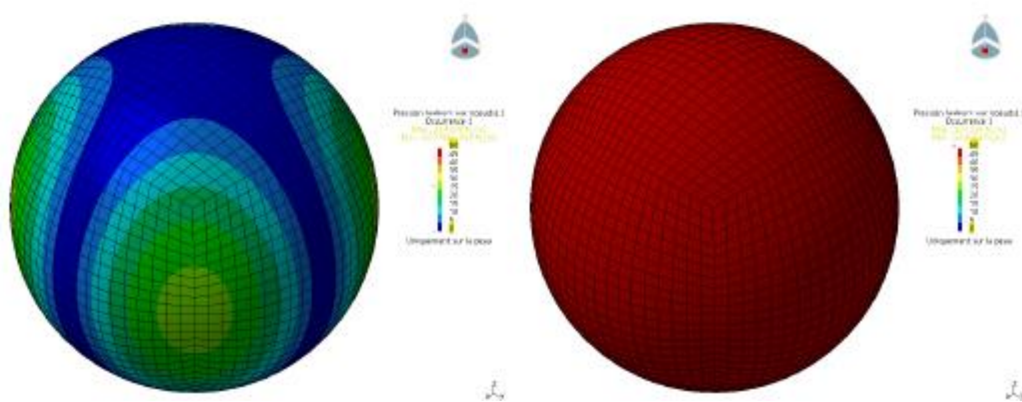




7.1: Left: one quadrupole $T_{xy} = 1$; right: one quadrupole $T_{xy} = 1.05$.



7.2: Left: two dipoles with amplitudes $D_{x,1-2} = \pm 200$ (left); right: $D_{x,1} = 210$, $D_{x,2} = 200$.



… use the corresponding Green's function!

A popular formulation for industrial applications

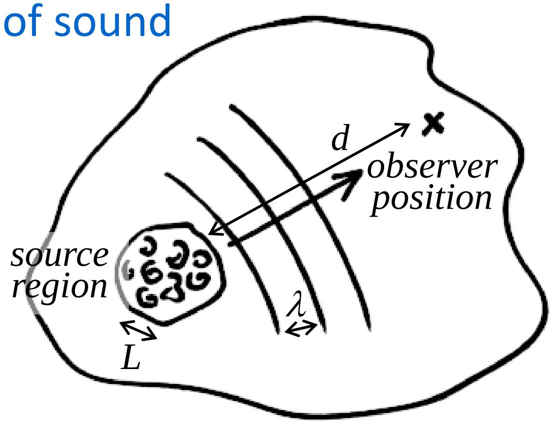
- Curle's formulation is quite powerful
  - It enforces the correct radiation pattern of each source component: $P_{\text{quadru}} / P_{\text{dipo}} \sim M^2$
  - At low Mach numbers, dipolar contribution dominates the quadrupolar one for compact sources
  - Surface scalar (p') data are much less demanding in memory than volumetric, tensorial (T_{ij}) data
  - Surface mesh often available from design stage

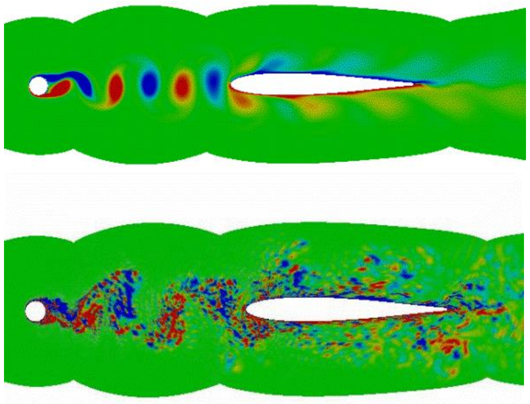
$$4\pi c_0^2\, \rho'(\mathbf{x},t) = \underbrace{\frac{\partial^2}{\partial x_i \partial x_j} \iiint_V \left[\frac{T_{ij}}{|\mathbf{x}-\mathbf{y}|}\right] \mathrm{d}^3\mathbf{y}}_{\text{Quadrupole, } W \propto M^8 \text{ in free field}} + \underbrace{\frac{\partial}{\partial x_i} \iint_{\partial V} \left[\frac{p' n_i}{|\mathbf{x}-\mathbf{y}|}\right] \mathrm{d}^2\mathbf{y}}_{\text{Dipole, } W \propto M^6 \text{ in free field}}$$

- BUT: tricky implementation for non-compact geometries…

The hybrid approach from a practical viewpoint

- The computation of flow is decoupled from the computation of sound
  - Acoustic prediction: post-processing of source field data
- Fundamental assumption: one-way coupling
  - Unsteady flow produces sound and affects its propagation
  - BUT: sound waves do not affect flow field significantly
  - Principal application of the hybrid approach: flows at low Mach numbers
- Usable simulation tools for the flow description:
  - Reynolds Averaged Navier-Stokes (RANS) solver → time-averaged data (SNGR to reconstruct transient) ☹
  - Unsteady RANS → unsteady, but only large scale 😐
  - Large Eddy Simulation (LES), Detached Eddy Simulation (DES) → unsteady, broadband turbulence (up to grid & scheme cut-off frequency) ☺
- Low-Mach number applications
  - Incompressible LES / DES solvers to reduce CPU cost
  - Careful interpretation of the flow data in aeroacoustical analogy

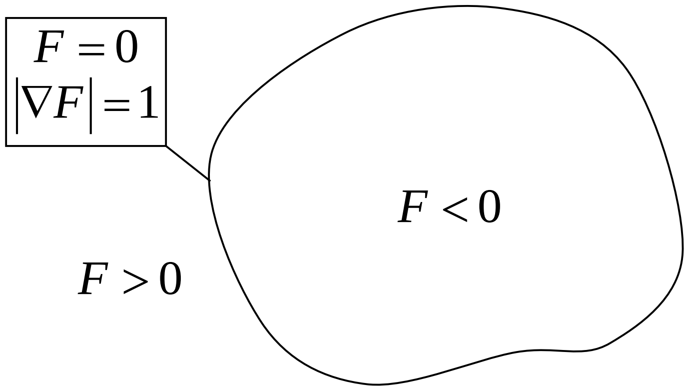




http://www.lmfa.ec-lyon.fr/recherche/turbo

Ffowcs Williams and Hawkings analogy: moving bodies

- Use of generalized functions to account for body motion
- Body motion trajectory described by function *F*(*x*,*y*,*z*,*t*):



- Heaviside and Dirac functions properties:

$$\begin{aligned} H(F) &= 0 && \text{inside solid body} \\ &= 1 && \text{in fluid region} \\ \nabla(H(F)) &= \delta(F)\,\nabla F \\ \frac{\partial}{\partial t}(H(F)) &= \delta(F)\,\frac{\partial F}{\partial t} \end{aligned}$$

Conservative equations and analogy using generalized functions

- Equations of conservation of mass and momentum can be rewritten as:

$$\frac{\partial(\rho' H)}{\partial t} + \frac{\partial(\rho v_i H)}{\partial x_i} = \rho_0\,(\mathbf{v}\cdot\nabla F)\,\delta(F) = \rho_0 V_n\,\delta(F)$$

$$\frac{\partial(\rho v_i H)}{\partial t} + \frac{\partial}{\partial x_j}\left[(\rho v_i v_j + p\,\delta_{ij} + \sigma_{ij})\,H\right] = (p\,\delta_{ij} + \sigma_{ij})\,\nabla F\,\delta(F)$$

- The analogy becomes:

$$\frac{\partial^2(\rho' H)}{\partial t^2} - c_0^2\,\frac{\partial^2(\rho' H)}{\partial x_i^2} = \frac{\partial^2}{\partial x_i \partial x_j}(T_{ij}H) - \frac{\partial}{\partial x_i}\left[(p\,\delta_{ij} + \sigma_{ij})\,\nabla F\,\delta(F)\right] + \frac{\partial}{\partial t}(\rho_0 V_n\,\delta(F))$$

Integral solution

- Using the free-field Green's function:

$$\rho'(\mathbf{x},t) = \frac{\partial^2}{\partial x_i \partial x_j} \int_{-\infty}^{t} \iiint_V \frac{\delta(t-\tau-|\mathbf{x}-\mathbf{y}|/c_0)}{4\pi c_0^2 |\mathbf{x}-\mathbf{y}|} T_{ij}(\mathbf{y},\tau)\,\mathrm{d}^3\mathbf{y}\,\mathrm{d}\tau$$

$$- \frac{\partial}{\partial x_i} \int_{-\infty}^{t} \iint_{\partial V} \frac{\delta(t-\tau-|\mathbf{x}-\mathbf{y}|/c_0)}{4\pi c_0^2 |\mathbf{x}-\mathbf{y}|} [p\,\delta_{ij} + \sigma_{ij}](\mathbf{y},\tau)\, n_j\,\mathrm{d}^2\mathbf{y}\,\mathrm{d}\tau$$

$$+ \frac{\partial}{\partial t} \int_{-\infty}^{t} \iint_{\partial V} \frac{\delta(t-\tau-|\mathbf{x}-\mathbf{y}|/c_0)}{4\pi c_0^2 |\mathbf{x}-\mathbf{y}|} \rho_0 V_n(\mathbf{y},\tau)\,\mathrm{d}^2\mathbf{y}\,\mathrm{d}\tau$$

- More practical to have the source described in a moving coordinate system *η* attached to the body.
- Retarded time equation: $g(t^*, t, \mathbf{x}, \boldsymbol{\eta}) \equiv t - t^* - \frac{|\mathbf{x} - \mathbf{y}(\boldsymbol{\eta}, t^*)|}{c_0} = 0$
- Dirac function property: $\int_{-\infty}^{\infty} \delta\left(h(\xi)\right)\, f(\xi)\,\mathrm{d}\xi = \sum_i \frac{f(\xi_i)}{|h'(\xi_i)|}, \quad h(\xi_i) = 0$

Doppler effects

- Applied to the quadrupolar component:

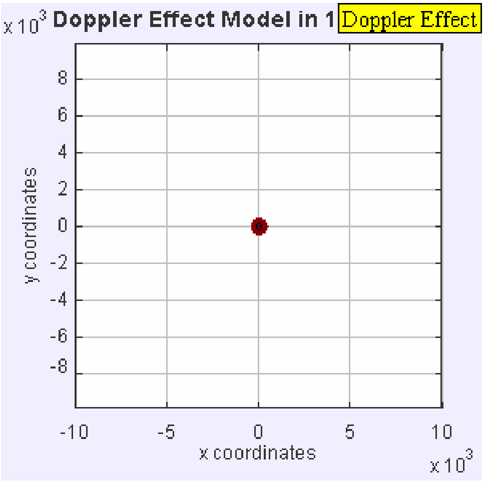
$$\int_{-\infty}^{t} \iiint_{V_\eta} \frac{\delta\left(g(\tau, t, \mathbf{x}, \boldsymbol{\eta})\right)}{4\pi c_0^2 |\mathbf{x} - \mathbf{y}(\boldsymbol{\eta}, \tau)|} T_{ij}\, \mathrm{d}^3\boldsymbol{\eta}\, \mathrm{d}\tau = \frac{1}{4\pi c_0^2} \iiint_{V_\eta} \left[\frac{T_{ij}}{R\,|1 - \mathbf{M}.\mathbf{R}/R|} \right] \mathrm{d}^3\boldsymbol{\eta}$$

$\mathbf{M}$: vector Mach number $\quad 1 - \mathbf{M}.\mathbf{R}/R$: Doppler factor

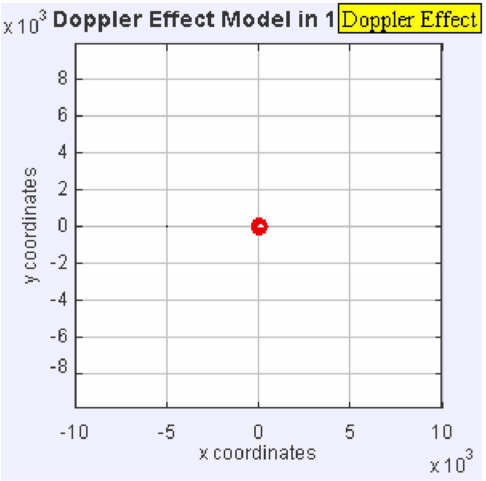
- Final solution: $$\rho'(\mathbf{x}, t) = \frac{1}{4\pi c_0^2} \frac{\partial^2}{\partial x_i \partial x_j} \iiint_{V_\eta} \left[\frac{T_{ij}}{R\,|1 - \mathbf{M}.\mathbf{R}/R|} \right] \mathrm{d}^3\boldsymbol{\eta} - \frac{1}{4\pi c_0^2} \frac{\partial}{\partial x_i} \iint_{\partial V_\eta} \left[\frac{(p\,\delta_{ij} + \sigma_{ij})\, n_j}{R\,|1 - \mathbf{M}.\mathbf{R}/R|} \right] \mathrm{d}^2\boldsymbol{\eta} + \frac{1}{4\pi c_0^2} \frac{\partial}{\partial t} \iint_{\partial V_\eta} \left[\frac{\rho_0 V_n}{R\,|1 - \mathbf{M}.\mathbf{R}/R|} \right] \mathrm{d}^2\boldsymbol{\eta}$$

- The flow intrinsic features (flow separation, turbulent transition, …) are expressed in the frame of reference attached to the moving axes.
- The Doppler effects (convective amplification and frequency shift) are given by the motion of the sources in the fixed coordinate system.

Effect of relative motion



Fixed monopole



Monopole convected at M = 0.7

(source: Wikipedia)

Doppler amplification

$$\Delta\varphi - \frac{1}{c_0^2}\frac{\partial^2\varphi}{\partial t^2} = S(\mathbf{x},t) \qquad S(\mathbf{y},\tau) = -q(\tau)\,\delta(\mathbf{y}-\mathbf{U}_0\tau)$$

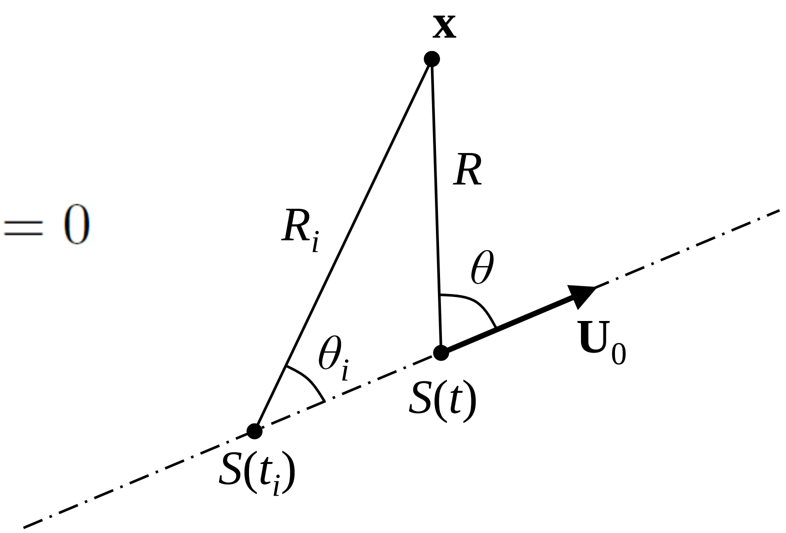
$$\varphi(\mathbf{x},t) = \int_{-\infty}^{\infty}\int_V q(\tau)\,\frac{\delta(\tau - t + |\mathbf{x}-\mathbf{y}|/c_0)}{4\pi|\mathbf{x}-\mathbf{y}|}\,\delta(\mathbf{y}-\mathbf{U}_0\tau)\,\mathrm{d}^3\mathbf{y}\,\mathrm{d}\tau$$

$$\int_{-\infty}^{\infty} f(\xi')\delta(g(\xi'))\,\mathrm{d}\xi' = \sum_j \frac{f(\xi_j)}{\left|\frac{\partial g}{\partial \xi}(\xi_j)\right|}$$

$$g(\tau) \equiv \tau - t + |\mathbf{x}-\mathbf{y}(\tau)|/c_0 = 0$$

$$\varphi(\mathbf{x},t) = \frac{1}{4\pi}\sum_j \frac{q(\tau_j)}{R_j\boxed{|1 - M_0\cos\theta_j|}}$$

Doppler factor:
amplitude modulation and directivity



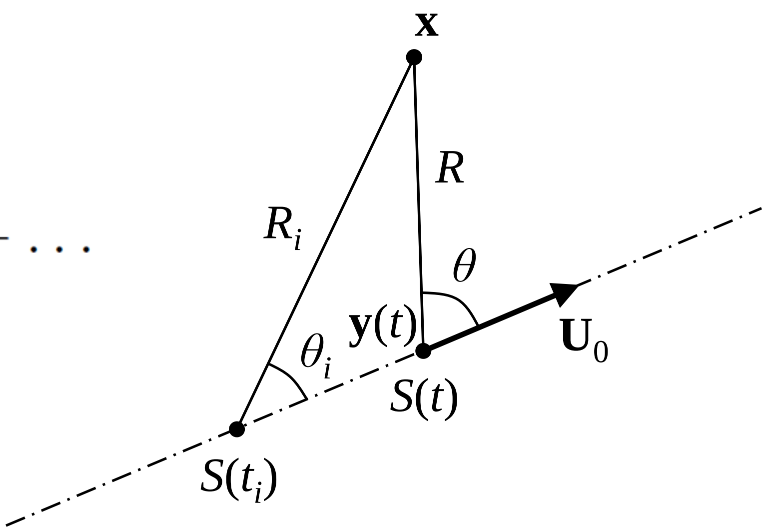
Doppler frequency shift

$$q(\tau) = \hat{q}\,\mathrm{e}^{\mathrm{i}\omega\tau}$$

$$\varphi(\mathbf{x},t) = \frac{\hat{q}}{4\pi}\frac{\mathrm{e}^{\mathrm{i}\omega(t-R_1(t)/c_0)}}{R_1(1-M_0\cos\theta_1)}$$

$$\frac{R_1(t)}{c_0} \sim \frac{R_1(t_e)}{c_0} + \frac{1}{c_0}\frac{\mathrm{d}R_1}{\mathrm{d}t}\bigg|_{t_e}(t-t_e)+\dots$$

$$\frac{\mathrm{d}R_1}{\mathrm{d}t} = -\frac{M_0\cos\theta_1}{1-M_0\cos\theta_1}$$



Phase factor : $\mathrm{e}^{\mathrm{i}\omega t/(1-M_0\cos\theta_1(t_e))}$

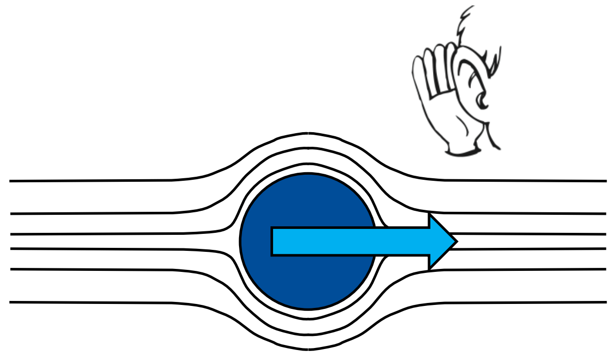
Modified frequency : $\dfrac{\omega}{\boxed{1-M_0\cos\theta_1(t_e)}}$

Doppler factor: frequency shift

The Doppler factor accounts for all amplitude modulation and frequency shift effects; the source unsteadiness can be described in a local (moving) coordinate system attached to the moving body.

Thickness, loading, turbulence noise

- Thickness noise:

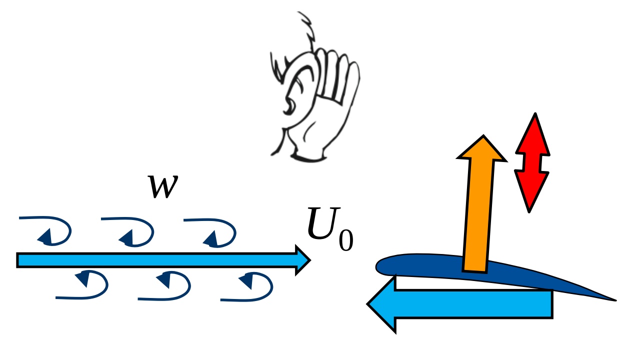


- Loading noise: steady and unsteady

$$\frac{\partial}{\partial \tau}\left(\frac{F_i}{D}\right) = \frac{1}{D}\frac{\partial F_i}{\partial \tau} - \frac{F_i}{D^2}\frac{\partial D}{\partial \tau} \qquad D = |1 - \mathbf{M}\cdot\mathbf{R}/R|$$

unsteady motion

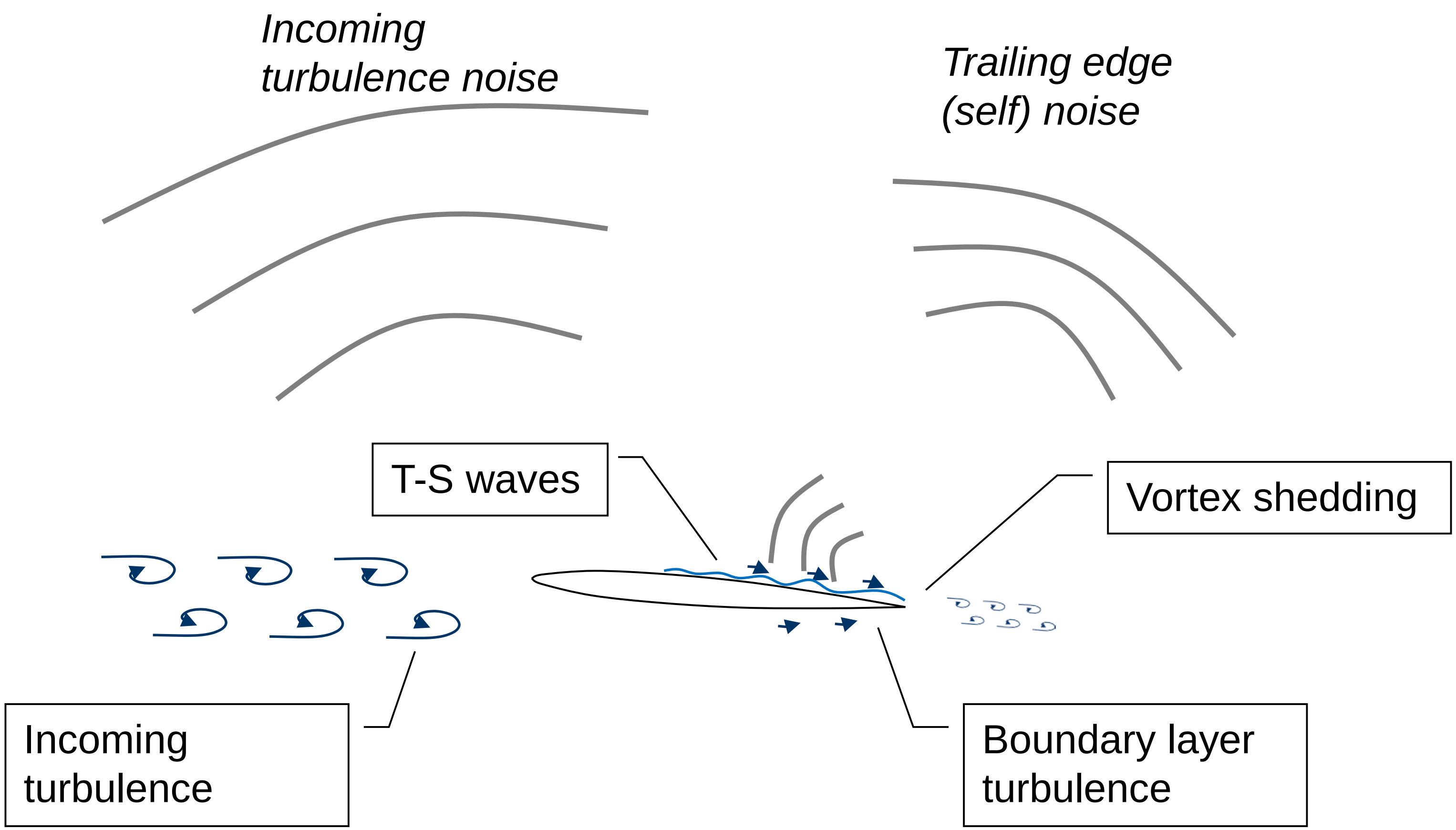
$$\frac{I_{\mathrm{motion}}}{I_{\mathrm{intrinsic}}} \sim \frac{1}{n^2}\left(\frac{U_0}{w}\right)^2\left(\frac{M_0}{1-M_0}\right)^2$$



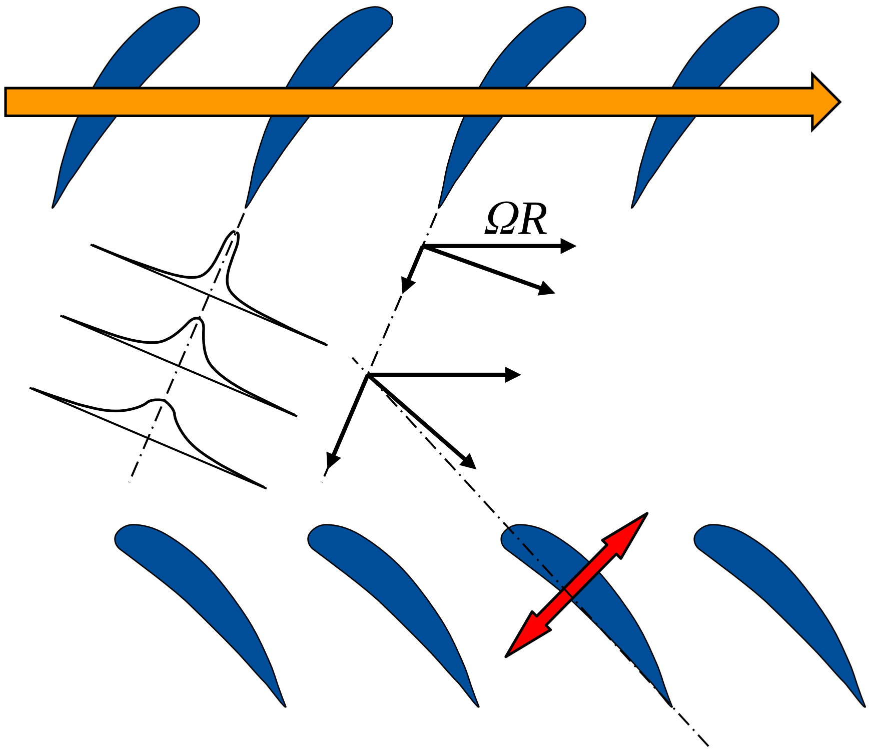
- Turbulence noise: same scaling, relative to unsteady loading noise, as for the motion loading noise → negligible as well

At low Mach numbers, and for thin rotor/stator blades/vanes, the unsteady loading noise dominates the sound production.

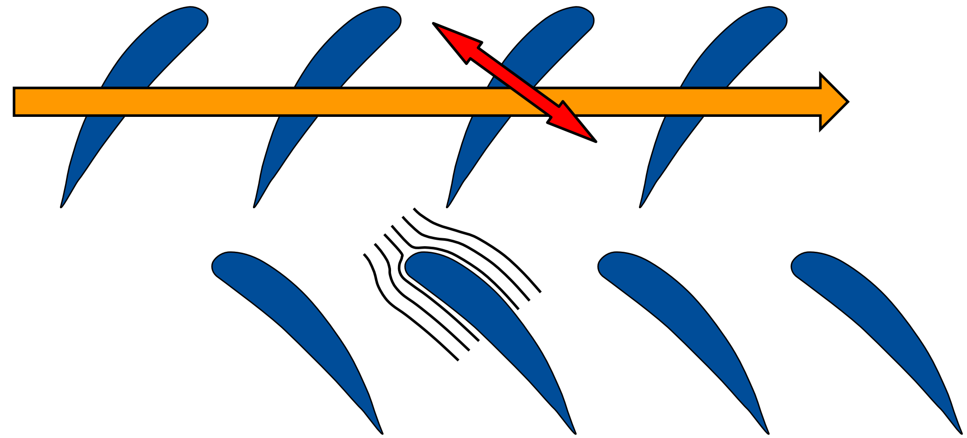
Loading noise mechanisms



Viscous and potential interactions

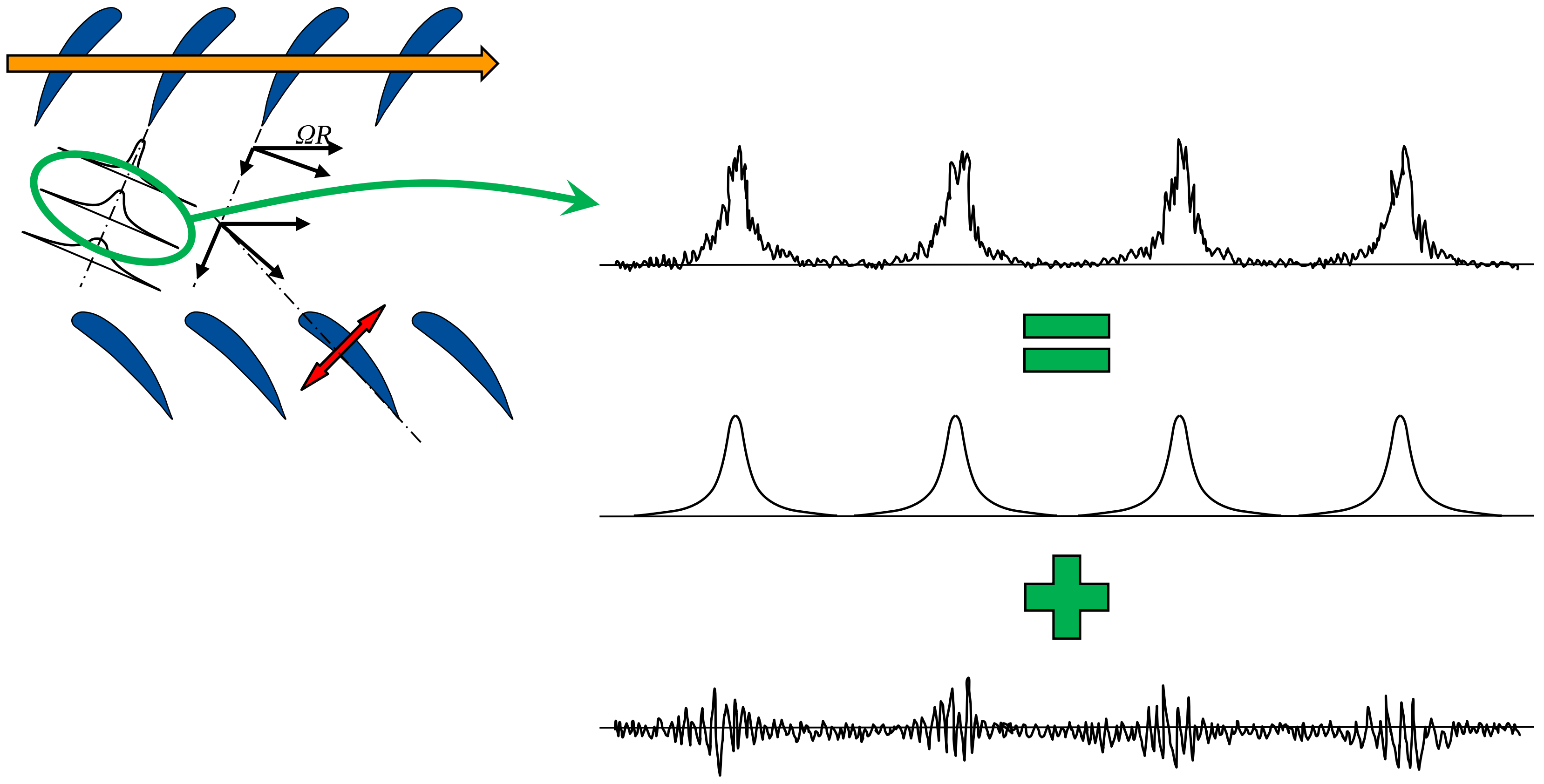


Stator vanes: varying inlet velocity angle and magnitude

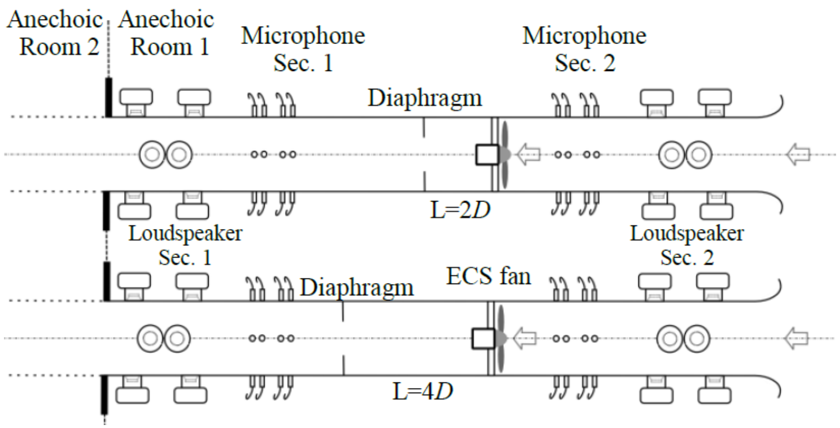


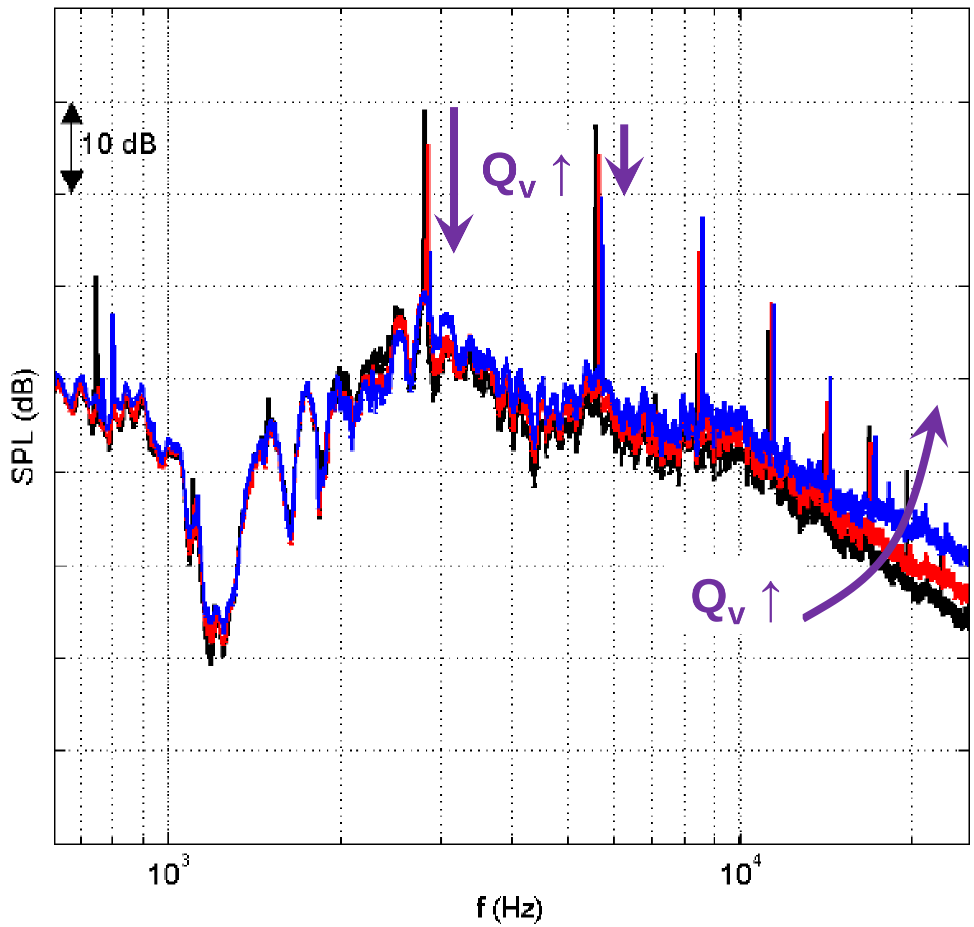
Rotor blades: varying outlet velocity angle and magnitude

Viscous interaction: tonal and broadband noise



Tonal and broadband noise





Kucukcoskun, K., Christophe, J., Schram, C., Holmberg, M., ISROMAC15, 2014.

Rotating point force

- FW-H analogy in time domain:

$$\rho'(\mathbf{x},t) = \underbrace{\int_{-\infty}^{t}\int_{S(\tau)} \frac{\partial G}{\partial y_i} f_i \,\mathrm{d}^2\mathbf{y}\,\mathrm{d}\tau}_{\textit{loading noise}} + \underbrace{\int_{-\infty}^{t}\int_{V(\tau)} \frac{\partial^2 G}{\partial y_i \partial y_j} T_{ij}\,\mathrm{d}^3\mathbf{y}\,\mathrm{d}\tau}_{\textit{quadrupolar noise}} + \underbrace{\int_{-\infty}^{t}\int_{S(\tau)} \frac{\partial G}{\partial \tau} \rho_0 V_n \,\mathrm{d}^2\mathbf{y}\,\mathrm{d}\tau}_{\textit{thickness noise}}$$

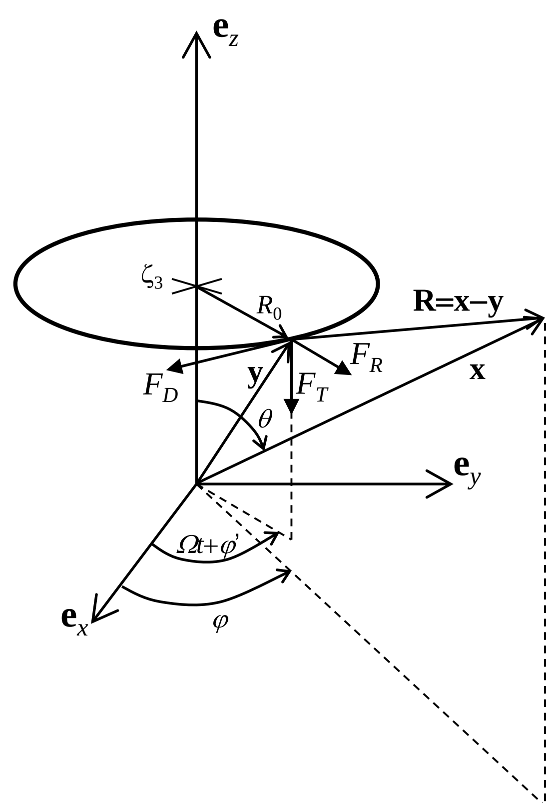
- Using the free field Green's function, and for a compact source:

$$\rho'(\mathbf{x},t) \sim -\frac{1}{4\pi c_0^2}\frac{\partial}{\partial x_i}\int_{S(\tau_0)}\left[\frac{f_i}{R|1-\mathbf{M}.\mathbf{R}/R|}\right]_{\tau^*}\mathrm{d}^2\zeta$$

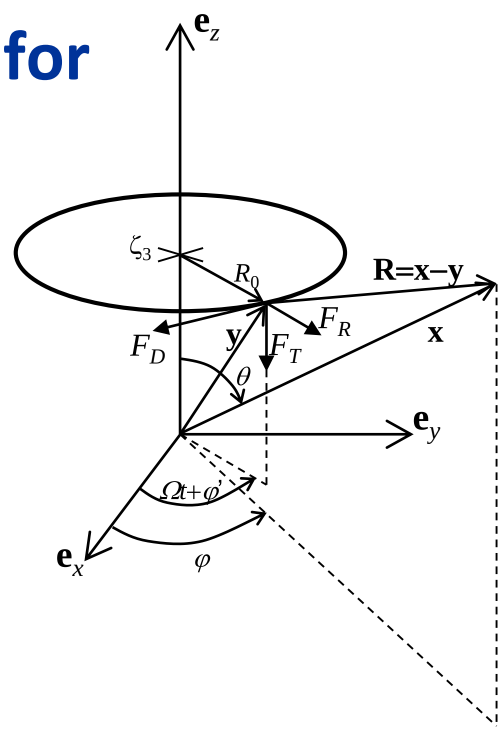
$$\sim -\frac{1}{4\pi c_0^2}\frac{\partial}{\partial x_i}\left[\frac{F_i}{RD}\right]$$

- In the Fourier domain:

$$\rho(\mathbf{x},\omega) = \frac{\mathrm{i}k}{8\pi^2 c_0^2}\int_{-\infty}^{\infty}\frac{\mathbf{F}.\mathbf{R}}{R^2}\left(1+\underset{\uparrow\ \textit{Near-field}}{\frac{1}{\mathrm{i}kR}}\right)\mathrm{e}^{-\mathrm{i}\omega(\tau+R/c_0)}\,\mathrm{d}\tau$$



Geometrical far-field approximation for a *B*-bladed axial rotor



Constructive interference: sound of the total fan = B x (sound of a single blade)

Bessel function: modulation of the Doppler frequency shift during blade revolution

$$\rho_{nB} \sim -\frac{\mathrm{i}Bk_{nB}}{4\pi c_0^2}\,\frac{\mathrm{e}^{-ik_{nB}x}}{x}\sum_{p=-\infty}^{+\infty}\mathrm{e}^{-i\,(nB-p)(\varphi-\pi/2)}\,\mathrm{J}_{-nB+p}(-k_{nB}R_0\sin\theta)\left[F_p^{(T)}\cos\theta-\frac{nB-p}{k_{nB}R_0}\,F_p^{(D)}\right]$$

Radius where force is applied

Sound emitted at BPFHs

Sum over BLHs

Drag harmonic

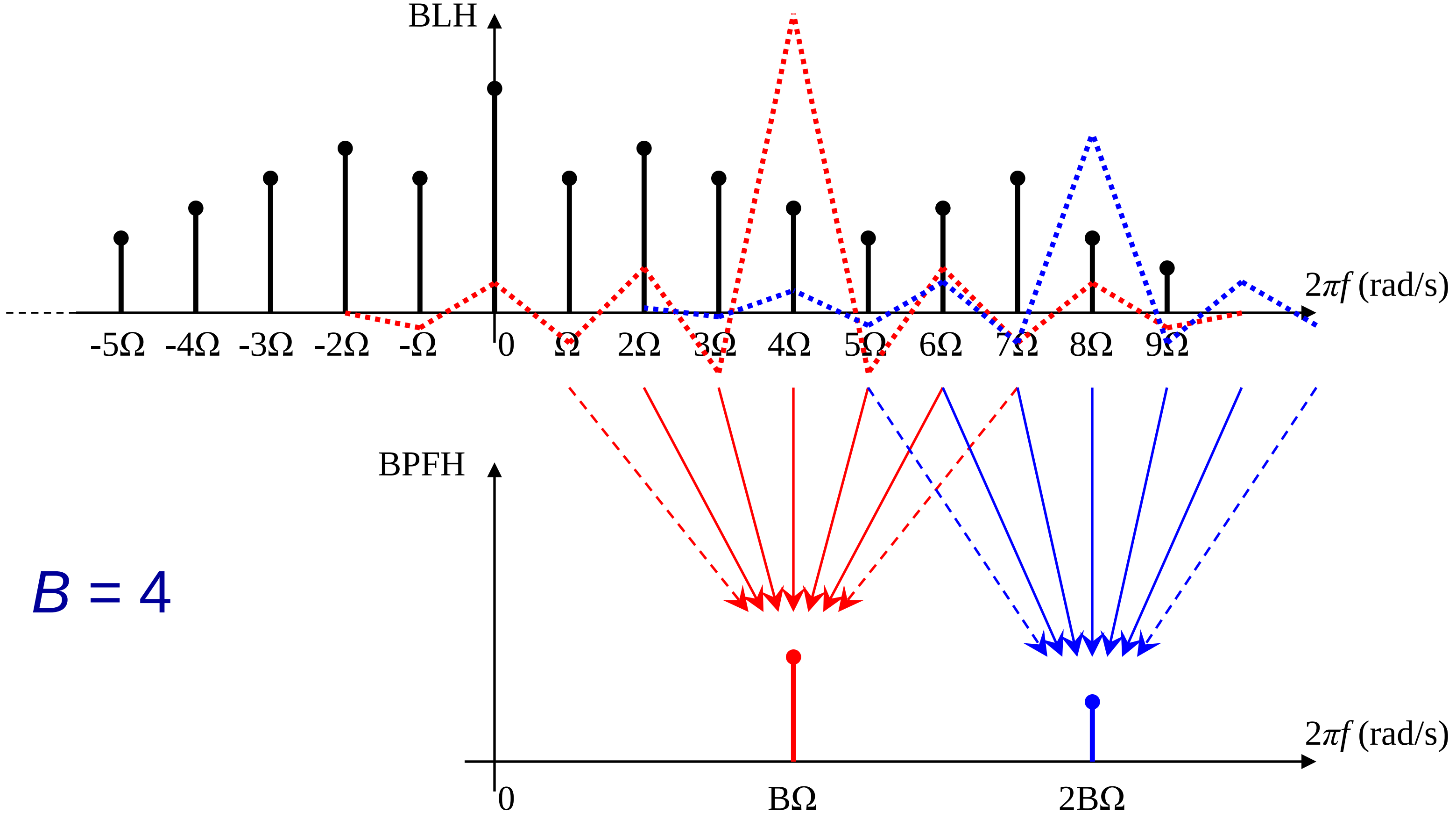
Listener polar angle

Listener azimuthal angle

Listener distance

Thrust harmonic

Doppler effect → summation over the BLHs



B = 4

$$\rho_{nB} \sim -\frac{\mathrm{i}Bk_{nB}}{4\pi c_0^2}\,\frac{\mathrm{e}^{-ik_{nB}x}}{x}\sum_{p=-\infty}^{+\infty} \mathrm{e}^{-i\,(nB-p)(\varphi-\pi/2)}\,\mathrm{J}_{-nB+p}(-k_{nB}R_0\sin\theta)\left[F_p^{(T)}\cos\theta - \frac{nB-p}{k_{nB}R_0}\,F_p^{(D)}\right]$$

Summary

- Aeroacoustical analogies allow extracting a maximum of acoustical information from a given description of the flow field

- Assuming a decoupling between the sound production and propagation, the analogies provide an explicit integral solution for the acoustical field at the listener position
  - Improves numerical robustness
  - Permits drawing scaling laws

- BUT: one has to make approximations and choices!
  - Acoustical variable (e.g. isothermal noise vs combustion noise)
  - Source term formulation (e.g. Lighthill's analogy vs Vortex Sound Theory)

- Some formulations make the dominant character of the source appear more explicitly, and allow making useful approximations.
- Without approximations, the analogy is useless!

A few references

- A. Pierce, *Acoustics: an Introduction to its Physical Principles and Applications*, McGraw-Hill Book Company Inc., New York, 1981.
- S.W. Rienstra and A. Hirschberg, *An Introduction To Acoustics* (corrections), Report IWDE 01-03 May 2001, revision every year or so…
- M.E. Goldstein, *Aeroacoustics*, McGraw-Hill International Book Company, 1976.
- A.P. Dowling and J.E. Ffowcs Williams, *Sound and Sources of Sound*, Ellis Horwood-Publishers, 1983.
- D.G. Crighton, A.P. Dowling, J.E. Ffowcs Williams, M. Heckl and F.G. Leppington, *Modern Methods in Analytical Acoustics*, Springer-Verlag London, 1992.

- And of course: the VKI Lecture Series…